

# Valuation of American Options via Basis Functions

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## Abstract

After a brief review of recent developments in the pricing and hedging of American options, this paper modifies the basis function approach to adaptive control and neuro-dynamic programming and applies it to develop (i) nonparametric pricing formulas for actively traded American options and (ii) simulation-based optimization strategies for complex over-the-counter options, whose optimal stopping problems are prohibitively difficult to solve numerically by standard backward induction algorithms because of the curse of dimensionality. An important issue in this approach is the choice of basis functions, for which some guidelines and their underlying theory are provided.

*Index Terms* — Optimal stopping, option pricing, function approximation, spline basis, neuro-dynamic programming.

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## I. INTRODUCTION

In the *New Palgrave Dictionary of Economics*, Ross [1] writes: “Despite such gaps (in the theory), when judged by its ability to explain the empirical data, option pricing theory is the most successful theory not only in finance, but in all of economics.” A call (put) option gives the holder the right to buy (sell) the underlying asset by a certain date  $T$  (known as the “expiration date” or “maturity”) at a certain price (known as the “strike price”). European options can be exercised only on the expiration date, whereas American options can be exercised at any time up to the expiration date. For European options, closed-form option pricing formulas were derived in the seminal papers of Black and Scholes [2] and Merton [3] through a dynamic hedging argument and a no-arbitrage condition. Except for Merton’s [3] result that American calls written on non-dividend-paying stocks reduce to European calls, American option valuation involves finite-horizon optimal stopping problems and has to be performed numerically. One of the “gaps” in option pricing theory noted by Ross is that “surprisingly little is known about the exercise properties” of American options. Another gap is related to “a surprisingly small empirical literature” which “should increase; options and option pricing theory give us an opportunity to measure directly the degree of anticipated uncertainty in the markets.”

During the fifteen years that have elapsed since Ross’ review, there have been many advances in option pricing theory and the gaps mentioned by Ross have been narrowed considerably, as will be reviewed in Section II. However, much still remains to be done. In this paper we modify the basis function approach to adaptive control and neuro-dynamic programming and apply it to theoretical and empirical analysis of American options. For actively traded American options, our approach in Section III uses market data to provide a sufficiently large training sample for estimating the parameters of a suitably chosen learning network, with which we then price and hedge a new option of similar type. The choice of basis functions for the learning network is based on a decomposition formula, given in Section II, for the value function of the optimal stopping problem that gives the price of an American option.

Section IV considers OTC (over-the-counter) American options. Unlike the situation in Section III, OTC options data are not available for nonparametric estimation of pricing formulas. We adopt the traditional parametric approach, but use basis functions to implement a simulation-based optimization scheme when the underlying optimal stopping problem is prohibitively difficult to solve numerically by backward induction because of the “curse

of dimensionality”. Tsitsiklis and Van Roy [4], [5] have recently introduced this “neurodynamic programming” approach to pricing highly complex American options that involve multidimensional asset prices, but have not provided a systematic way of choosing the basis functions for this approach. In Section IV we consider this problem, review recent developments in simulation-based optimization for American option valuation and provide some underlying theory. The interplay of leading-edge stochastic control theory and American option valuation is discussed in Sections IIIA and IVB, and some concluding remarks are given in Section V.

## II. OPTIMAL STOPPING PROBLEMS AND A DECOMPOSITION FORMULA

For a given strike price  $K$ , the payoff of a standard American option is  $g(S_t)$ , where  $S_t$  is the asset price at exercise time  $t$  and  $g(S) = (K - S)_+$  or  $(S - K)_+$  for a put or call, where  $x_+ = \max(x, 0)$ . Under the Black-Scholes assumptions on the market environment and the dynamics of the asset price process  $S_t$  (see Section IIIA), the price of an American option at time  $t \in [0, T]$  is given by

$$V(t, S) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-r(\tau-t)} g(S_\tau) | S_t = S], \quad (1)$$

where  $\mathcal{T}_{t,T}$  is the set of stopping times taking values between  $t$  and  $T$  and expectation is taken with respect to the risk-neutral measure under which

$$S_t = S_0 \exp\{(r - d - \sigma^2/2)t + \sigma B_t\} \quad (2)$$

is a geometric Brownian motion. In (2),  $\{B_t, t \geq 0\}$  is a standard Brownian motion,  $r$  is the interest rate (or rate of return of a risk-free asset such as Treasury bill),  $d$  stands for the dividend rate paid by the asset and  $\sigma$  is the volatility (or standard deviation of the asset’s return).

The stopping (early exercise) region  $\mathcal{E}$  for the optimal stopping problem (1) consists of all  $(t, S)$  with  $0 \leq t \leq T$  such that  $S \leq b(t)$  for puts and  $S \geq b(t)$  for calls, where  $b(\cdot)$  is a monotone continuously differentiable function except for the case of a call with  $d = 0$  (no dividends); see [6]. The pair  $(U(t, S), b(t))$  satisfies the free boundary PDE:

$$\frac{\partial U}{\partial t} + (r - d) \frac{\partial U}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} = rU \quad \text{for } (t, S) \notin \mathcal{E}, \quad (3a)$$

$$U(t, S) = g(S) \quad \text{and} \quad \partial U / \partial S = g'(S) \quad \text{for } S = b(t). \quad (3b)$$

In (3b),  $g'(S) = 1$  (or  $-1$ ) for calls (or puts). Finite difference methods to solve (3) numerically for both  $U(t, S)$  and  $b(t)$  dated back to Brennan and Schwartz [7]; subsequent refinements include [8] and [9].

Cox, Ross and Rubinstein [10] proposed to approximate the geometric Brownian motion (2) by a binomial tree, with root node  $S_0$  at time 0, to which dynamic programming via backward induction can be applied to solve the corresponding discrete-time and discrete-state optimal stopping problem. This approach has been widely used to compute the option price but it does not give the early exercise boundary. To compute both option values and the early exercise boundary, AitSahlia and Lai [6] recently proposed a modification that approximates the Brownian motion  $B_t$  by a symmetric random walk with time increment  $\delta$  and space increment  $\sqrt{\delta} X_i$ , where  $P(X_i = 1) = \frac{1}{2} = P(X_i = -1)$ . Noting that  $B_{\sigma^2 t}/\sigma$  has the same distribution as  $B_t$ , they first used the change of variables

$$\rho = r/\sigma^2, \quad \theta = d/r; \quad u = \sigma^2(t - T), \quad z = \log(S/K) - (\rho - \theta\rho - 1/2)u \quad (4)$$

so that the boundary  $b(t)$  becomes  $\bar{z}(u)$  in the new coordinate system and the payoff function  $g(S)$  becomes  $Kh(u, z)$ . The optimal stopping problem (1) can be expressed as  $Ke^{\rho u}\omega(u, z)$ , where

$$\omega(u, z) = \sup_{\tau \in \mathcal{T}_{u,0}} E[e^{-\rho\tau} h(\tau, B_\tau) | B_u = z]. \quad (5)$$

Note that the optimal stopping problem in (5) has terminal time 0, and that  $u = -\sigma^2 T$  at  $t = 0$ . For usual values of  $\sigma$  (between .1 and .4),  $\sigma^2 T$  is only a small fraction of the horizon  $T$  (measured in years). The Bernoulli walk approximation to  $B_u$  leads to the approximation of (5) by the backward recursion

$$\omega(u_i, z) = \max\{e^{-\rho u_i} h(u_i, z), \frac{1}{2}[\omega(u_{i-1}, z + \sqrt{\delta}) + \omega(u_{i-1}, z - \sqrt{\delta})]\}, \quad (6)$$

with  $u_0 = 0$ ,  $u_j = u_{j-1} - \delta$  and  $z \in Z_\delta \triangleq \{0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}, \dots\}$ . Each point in  $Z_\delta$  can be determined to be a stopping (or continuation) point at time  $u_i$  according as  $e^{-\rho u_i} h(u_i, z) =$  (or  $>$ )  $\omega(u_i, z)$ . A continuity correction to compute  $\bar{z}(u_i)$  from the discrete-time optimal stopping boundary  $\bar{z}_\delta(u_i)$  and interpolation schemes to compute  $\omega(u, z)$  for  $u \notin \{u_0, u_1, \dots\}$  and  $z \notin Z_\delta$  are given in [6]. The numerical results reported in [6] show that  $\bar{z}(\cdot)$  can be well approximated by a linear spline with a few knots.

#### A. Decomposition Formula and Its Applications

Jacka [11] and Carr, Jarrow and Myneni [12] derived the following decomposition of the price  $P(t^*, S)$  of an American put into a European put  $p(t^*, S)$  plus an integral which corresponds to the “early exercise premium”:

$$P(t^*, S) = p(t^*, S) + K\rho e^{\rho u} \int_u^0 \left\{ e^{-\rho s} \Phi\left(\frac{\bar{z}(s)-z}{\sqrt{s-u}}\right) - \theta e^{-(\theta\rho s+u/2)+z} \Phi\left(\frac{\bar{z}(s)-z}{\sqrt{s-u}} - \sqrt{s-u}\right) \right\} ds, \quad (7)$$

where  $t^* = T - t$ ,  $\Phi(x)$  denotes the standard normal distribution function. There is a corresponding formula for American call prices  $C(t^*, S)$  when  $d > 0$ ; see [6]. Black and Scholes [2] have derived closed-form expressions for European puts  $p(t^*, S)$  and calls  $c(t^*, S)$ :

$$p(t^*, S) = K e^{-rt^*} \Phi(-d_2(S, K, t^*)) - S e^{-dt^*} \Phi(-d_1(S, t^*)), \quad (8)$$

$$c(t^*, S) = S e^{-dt^*} \Phi(d_1(S, K, t^*)) - K e^{-rt^*} \Phi(d_2(S, K, t^*)), \quad (9)$$

where  $d_1(x, y, v) = \{\log(x/y) + (r - d + \sigma^2/2)v\}/\sigma\sqrt{v}$ ,  $d_2(x, y, v) = d_1(x, y, v) - \sigma\sqrt{v}$ .

Putting  $S = b(t)$  in (7) leads to an integral equation for  $b(t)$ , or equivalently, for  $\bar{z}(u)$ . Huang, Subrahmanyam and Yu [13] proposed to approximate the integrands in this integral equation and also in (7) by piecewise constant functions, with  $n = 1, 2$ , and 3 pieces, yielding three crude approximations  $P_1, P_2$ , and  $P_3$  to the option price, and then to combine them via a three-point Richardson extrapolation scheme. Instead of using piecewise constant approximations to the integrands, Ju [14] proposed to approximate the boundary by a piecewise exponential function for which the integral in (7) can be evaluated *in closed form*; see [14], [15]. Note that a piecewise exponential function for  $b(t)$  is equivalent to a piecewise linear function for  $\bar{z}(u)$ , and the numerical results in [6] show that  $\bar{z}$  is indeed well approximated by a piecewise linear function with a few pieces. Ju [14] reported numerical studies showing his method with 3 equally spaced pieces substantially improves previous approximations to option prices in both accuracy and speed. Refinements of Ju’s method were subsequently introduced by AitSahlia and Lai [6], [15].

The decomposition formula (7) can also be applied to compute the hedge parameter  $\Delta = \partial P/\partial S$ . The PDE (3a) is related to the no-arbitrage condition that the risk-free portfolio consisting of  $-1$  unit of the option and  $\Delta$  units of the asset should have the same rate of return as the risk-free asset; see [16, pp. 246-247]. This is the basis of “delta hedging” prior to the exercise of an American option. As shown in [15, pp. 94-96], the integral obtained by differentiating that in (7) with respect to  $S$  also has a closed-form expression when  $\bar{z}(\cdot)$  is piecewise linear, and approximating  $\bar{z}(\cdot)$  by a linear spline that uses unevenly spaced knots gives a fast and reasonably accurate method for computing  $\Delta$ .

## B. Extensions to Exotic Options

Chapter 18 of [16] gives an overview of *exotic options*, whose payoffs are more complicated than standard calls and puts and which are mostly traded in the over-the-counter market. Analogous to (1), the price of an American exotic option is again given by an optimal stopping problem (cf. [17]). The more complicated payoff, however, introduces much greater complexity into the optimal stopping problem.

For example, for American barrier options, the payoff is  $g(S_\tau)1_{\{\tau < \tau_H\}}$ , where  $H$  is an up-and-out barrier and  $\tau_H = \inf\{t \leq T : S_t \geq H\}$ . Boyle and Lau [18] pointed out that naive application of the Cox-Ross-Rubinstein binomial tree method can result in significant errors even when a large number of time steps is used because the barrier typically lies between two adjacent layers of nodes in the lattice. They proposed to reduce the size of errors by refining the time partition so that the resulting lattice has layers as close as possible to the barrier. AitSahlia, Imhof and Lai [19] modified the Bernoulli walk method described in (4)-(6) above to address this difficulty. Using the change of variables

$$z = \log(S/K), \quad \gamma = \log(H/K), \quad (10)$$

and defining  $\rho, \theta$  and  $u$  as in (4), they express the option price as  $Ke^{\rho u}w(u, z)$ , where

$$w(u, z) = \sup_{\tau \in \mathcal{T}_{u,0}} E[e^{-\rho\tau}(1 - e^{W_\tau})_+ 1_{\{\tau < \tau(\gamma)\}} | W_u = z] \quad (11)$$

for puts, with  $(1 - e^{W_\tau})_+$  replaced by  $(e^{W_\tau} - 1)_+$  for calls. Here  $\{W_u, u \leq 0\}$  is a Wiener process with drift  $\lambda = \rho - \theta\rho - \frac{1}{2}$ , and  $\tau(\gamma) = \inf\{s \leq 0 : W_s \geq \gamma\}$ . For  $\delta > 0$ , let  $u_0 = 0$  and  $u_k = u_{k-1} - \delta$ . To solve the optimal stopping (11), they approximate  $W_{u_k}$  by a Bernoulli random walk  $\sum_{i=1}^k X_i$  on the lattice  $L_\delta \triangleq \{\gamma - \sqrt{\delta}(1 + \delta\lambda^2)^{\frac{1}{2}}j : j = 0, 1, 2, \dots\}$ . The Bernoulli random walk has absorbing barrier  $\gamma \in L_\delta$ , time increment  $\delta$  and space increment  $X_i$  such that

$$P\{X_i = \pm \sqrt{\delta}(1 + \delta\lambda^2)^{\frac{1}{2}}\} = \frac{1}{2} \left( 1 \pm \frac{\lambda\sqrt{\delta}}{\sqrt{1 + \delta\lambda^2}} \right).$$

This leads to the backward recursion  $w(u_i, z) = \max\{e^{-\rho u_i}(1 - e^z)_+, Ew(u_{i-1}, z + X_n)\}$ , analogous to (6), which can be used to compute option values and the early exercise boundary  $\bar{z}(\cdot)$  (from  $\bar{z}_\delta(\cdot)$  via a continuity correction). Numerical results in [19] show that  $\bar{z}(\cdot)$  is well approximated by a linear spline with a few knots.

Analogous to (7), the price of an American up-and-out barrier option can be represented as a sum of the corresponding European barrier option (which has a closed-form expression)

and an early exercise premium which can be represented as an integral that involves  $\bar{z}(\cdot)$ . Moreover, the integral can be evaluated in closed form if  $\bar{z}(\cdot)$  is piecewise linear. Making use of this, [19] extends Ju's [14] method and its refinements in [6] and [15] to American barrier options.

American exotic options may lead to optimal stopping problems involving multidimensional state variables. A case in point is the lookback option whose payoff is  $g(M_t)$ , where  $M_t = \max_{0 \leq u \leq t} S_u$  for calls and  $M_t = \min_{0 \leq u \leq t} S_u$  for puts. The state vector at time  $t$  is  $(S_t, M_t)$  and the value function is of the form

$$U(t, S, M) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-r(\tau-t)} g(M_\tau) | S_t = S, M_t = M].$$

Using the change of variables (4) and  $z^* = \log(M/K)$ , Lai and Lim [20] extended the Bernoulli walk method in (4)-(6) to compute the option values and the early exercise boundary  $\bar{z}(u, z^*)$  of an American lookback option. Their numerical results show that  $\bar{z}(\cdot, z^*)$  can be well approximated by a linear spline with a few knots for every fixed  $z^*$  belonging to a grid. They also proved a decomposition formula, analogous to (7), that expresses  $U(t, S, M)$  as a sum of the European price (which has a closed-form expression) plus an early exercise premium which is an integral involving the early exercise boundary  $\bar{z}(u, z^*)$ , and made use of this to develop two fast approximate valuation methods.

Note that for the two-dimensional state variable  $(S_t, M_t)$ , the dynamics of  $M_t$  is induced by that of the process  $\{S_u, u \leq t\}$ , so there is only one underlying stochastic process driving the state vector. This provides considerable simplification and makes backward induction still feasible. For payoffs associated with multidimensional stochastic variables, the situation is more complicated and the "curse of dimensionality" makes backward induction of the type described above prohibitively difficult. In Section IV we address this problem by using a simulation-based approach that involves suitably chosen basis functions.

### III. NONPARAMETRIC VALUATION USING OPTIONS DATA

In the past decade, a number of papers have proposed to use nonparametric methods for option pricing, beginning with the work of Hutchinson, Lo and Poggio [21] on European options. The nonparametric pricing method of [21] uses a "learning network", which is trained from historical data on option prices, to provide a formula relating the output (i.e., option price) to an input vector  $x$  consisting of the time  $t^*$  to maturity and the ratio of the current stock price  $S$  to the strike price  $K$ . Three kinds of networks  $Kf(x)$  are considered in [21], namely,

- (i) RBF networks  $f(x) = \beta_0 + \alpha^T x + \sum_{i=1}^I \beta_i h_i(\|A(x - \gamma_i)\|)$ , where  $A$  is a positive definite matrix and  $h_i$  is a radial basis function (RBF) of the form  $e^{-y^2/\sigma_i^2}$  or  $(y^2 + \sigma_i^2)^2$ ;
- (ii) neural networks  $f(x) = \psi(\beta_0 + \sum_{i=1}^I \beta_i h(\gamma_i + \alpha_i^T x))$ , where  $h(y) = 1/(1 + e^{-y})$  is the logistic function and  $\psi$  is either the identity function or the logistic function;
- (iii) PPR networks  $f(x) = \beta_0 + \sum_{i=1}^I \beta_i h_i(\alpha_i^T x)$ , where  $h_i$  is an unspecified function that is estimated from the data by projection pursuit regression (PPR).

The  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  above are unknown parameters of the network that are to be estimated from the data. As pointed out in [21], all three classes of networks have some form of “universal approximation property”. A simulation study in [21] shows that these learning networks trained from a two-year sample of daily option prices can approximate the Black-Scholes formula well and can be used successfully to both price and delta-hedge options not in the sample.

Instead of using these learning networks to approximate the option price, Broadie et al. [22] used kernel smoothers to estimate the option pricing formula of an American option. Using a training sample of daily closing prices of American calls on the S&P100 Index that were traded on the Chicago Board Options Exchange from 3 January 1984 to 30 March 1990, they compared the nonparametric estimates of American call option prices at a set of  $(S/K, t^*)$  values with those obtained by approximations due to Broadie and Detemple [23]. They reported statistically significant differences between the parametric and nonparametric estimates.

#### A. Connection with Stochastic Adaptive Control

It may be illuminating to consider the parametric *vis-a-vis* nonparametric approaches to American option pricing from the viewpoint of stochastic adaptive control. Economic theory (involving no-arbitrage equilibrium and reward maximization) leads to the optimal stopping problem (1) for the price and early exercise region of an American option, under the Black-Scholes assumptions of no transaction costs, no limits on short selling, continuous trading, constant interest rates and a geometric Brownian motion (with constant volatility) for the asset price. The optimal stopping problem based on these assumptions involves several parameters: volatility  $\sigma$ , interest rate  $r$ , strike price  $K$ , current asset price  $S$  and time  $t^*$  to maturity, assuming no dividend payments for simplicity. Of these,  $K$  and  $t^*$  are specified in the contract and  $S$  is the quoted asset price on the contract’s starting date. The volatility  $\sigma$  and interest rate  $r$  have to be estimated in some way from historical data on

asset prices and the US Federal funds target rate or 1 month Treasury bill yields. How to estimate them well is, therefore, part of the adaptive control problem (i.e., optimal stopping with unknown parameters). Since the underlying asset price process  $S_t$  is assumed to be geometric Brownian motion,  $\sigma$  can be estimated from the training sample of asset prices by maximum likelihood, as is usually done in econometric studies (e.g. [22]). However, it is widely recognized in the financial industry that volatility changes with time and therefore moving averages are often used to estimate  $\sigma$ ; see Chapter 15 of [16]. Likewise  $r$  is also a time-varying parameter, which is incompatible with using the average of a long historical series of daily returns of 1 month Treasury bills as in [22]. Stochastic adaptive control with possibly time-varying parameters provides useful insights into the conventional parametric approach to hedging and pricing American options, as will be demonstrated in Section III C.

Neural and other learning networks have a burgeoning literature in adaptive control of nonlinear dynamic systems for which an explicit parameterization of the control system is either unknown or intractable. The nonparametric approach uses these networks for approximating the input-output relationship in a massively parallel computing framework (cf. [24], [25]). In particular, Sanner and Slotine [25] use RBF networks to develop “direct” adaptive controllers for which there is no explicit attempt to model the system dynamics. The controller tunes the adjustable parameters of the network in response to observed deviations of the system from its desired behavior. A complication in the design of a globally stable control scheme with this approach is that since the RBF network consists of only a finite number of nodes, its approximation capabilities can be guaranteed only on a subset of the plant state space, so one needs an additional control component which takes over the adaptive component and forces the state back into the “good” subset when the approximation ability begins to deteriorate; see [25]. The adaptive control problem for nonparametric valuation of American options is considerably simpler because learning is based not on the option to be priced, but on many other similar options. In other words, the training data are exogenous rather than endogenous. Moreover, the control problem is an optimal stopping problem for which stability is no longer an issue. Nevertheless, the idea of training a network that approximates the control system via basis functions instead of modeling the systems dynamics is also central to the nonparametric approach to American option valuation.

### *B. Basis Functions for the Nonparametric Approach*

For European calls, Hutchinson, Lo and Poggio [21] assume a price of the form  $Kf(S/K, t^*)$  and propose three classes of basis functions to approximate  $f$ , as described above. Since the

Black-Scholes formula has proved to be quite successful in explaining empirical data (see Section I), an alternative approach is to express the option price as  $c + Ke^{-rt^*} f^*(S/K, t^*)$ , where  $c$  is the Black-Scholes price (9). This is tantamount to including  $c(t, S)$  as one of the basis functions (with prescribed weight 1) to come up with a more parsimonious approximation to the actual option price.

The usefulness of this idea is even more apparent in the case of American options. Focusing on puts for definiteness, the results of Section IIA show that an American put can be expressed as the sum of a European put  $p$  (given in (8)) and a remainder (the early exercise premium) which is typically small relative to  $p$ . This suggests that  $p$  should be included as one of the basis functions (with prescribed weight 1). We propose to use additive regression splines after the change of variables  $u = -\sigma^2 t^*$  and  $z = \log(S/K)$ . Specifically, for small  $t^*$  (say within 5 trading days prior to expiration, i.e.  $t^* \leq 5/253$  under the assumption of 253 trading days per year), we approximate  $P$  by  $p$ . For  $t^* > 5/253$  (or equivalently,  $u < -5\sigma^2/253$ ), we approximate  $P$  by

$$P = p + Ke^{\rho u} \left\{ \alpha + \alpha_1 u + \sum_{j=1}^{J_u} \alpha_{1+j} (u - u^{(j)})_+ + \beta_1 z + \beta_2 z^2 + \sum_{j=1}^{J_z} \beta_{2+j} (z - z^{(j)})_+^2 + \gamma_1 w + \gamma_2 w^2 + \sum_{j=1}^{J_w} \gamma_{2+j} (w - w^{(j)})_+^2 \right\}, \quad (12)$$

where  $\rho = r/\sigma^2$  as in (4),  $\alpha$ ,  $\alpha_j$ ,  $\beta_j$  and  $\gamma_j$  are regression parameters to be estimated by least squares from the training sample and

$$w = |u|^{-1/2} \{z - (\rho - \theta\rho - 1/2)u\} \quad (\theta = d/r) \quad (13)$$

is an ‘‘interaction’’ variable derived from  $z$  and  $u$ . The motivation behind the centering term  $(\rho - \theta\rho - 1/2)u$  comes from the Brownian coordinates (4), whereas that behind the normalization  $|u|^{-1/2}$  comes from (7) and the closely related  $d_1(x, y, v)$  in (9). The knots  $u^{(j)}$  (respectively  $z^{(j)}$  or  $w^{(j)}$ ) of the linear (respectively quadratic) spline in (12) are the  $100j/J_u$  (respectively  $100j/J_z$  and  $100j/J_w$ )-th percentiles of  $\{u_1, \dots, u_n\}$  (respectively  $\{z_1, \dots, z_n\}$  or  $\{w_1, \dots, w_n\}$ ). The choice of  $J_u$ ,  $J_z$  and  $J_w$  is over all possible integers between 1 and 10 to minimize the generalized cross validation (GCV) criterion, which can be expressed in the following form (cf. [26], [27]):

$$\text{GCV}(J_u, J_z, J_w) = \sum_{i=1}^n (P_i - \hat{P}_i)^2 \left/ \left\{ n \left( 1 - \frac{J_u + J_z + J_w + 6}{n} \right)^2 \right\} \right.,$$

where  $P_i$  and  $\hat{P}_i$  are the observed and fitted (via (12)) American option prices, respectively.

In the preceding we have assumed prescribed constant  $\gamma$  and  $\sigma$  as in the Black-Scholes model; these parameters appear in (12) via the change of variables (4). In practice  $\sigma$  is unknown and may also vary with time. We can replace it in (12) by the standard deviation  $\hat{\sigma}_t$  of the most recent asset prices, say, during the past 60 trading days prior to  $t$  as in [21, p. 881]. This is tantamount to incorporating the asset prices  $S_{t-1}, \dots, S_{t-60}$  in the formula  $P(t, S; S_{t-1}, \dots, S_{t-60})$  with  $S_t = S$ . Moreover, the risk-free rate  $r$  may also change with time, and can be replaced by the yield  $\hat{r}_t$  of a short-maturity Treasury bill on the close of the month before  $t$ . The same remark also applies to the dividend rate.

In [21] and [22], the moving-window estimates  $\hat{\sigma}_t$  and  $\hat{r}_t$  are used only in the parametric (Black-Scholes/Broadie-Detemple) formula but not in the nonparametric pricing formulas. When  $\sigma$  and  $r$  are assumed to be constant, as in the Black-Scholes model, the nonparametric approach does not need to incorporate them into the learning network for the pricing formula. However, the 5-year period from January 1987 to December 1991 used in the empirical study in [21] and the 6-year study from January 1984 to March 1990 in [22] both include the October 1987 market crash and experience substantial changes in volatility and interest rates. In contrast to [21] and [22], our nonparametric pricing formula incorporates these basic time-varying parameters in a parsimonious way that is suggested by economic theory.

For European options, Hutchinson, Lo and Poggio [24] proposed to use neural networks, or RBF or PPR networks, as approximations to the prices. Although the basis functions in these networks can also be used for American options and have certain advantages (e.g., universal approximation bounds that do not depend on the dimensionality of the predictor variable  $x$ ; see [28]), estimation of the network parameters involves computationally intensive nonlinear optimization procedures. By using additive regression models involving splines as basis functions, we can use linear least squares, which have much lower computational complexity than the preceding networks, to estimate the parameters of (12). Moreover, we can use the relatively simple GCV criterion to choose the number of knots of the splines. Friedman and Silverman [29] have given a detailed discussion of the computational and statistical advantages of these additive spline models in nonparametric regression with multivariate predictor variables  $x$ , and have also provided refinements of the knot placement procedure described above. They have also pointed out the relationship between spline (piecewise polynomial) smoothers and kernel (local averaging) smoothers for univariate predictors  $x$ ; see also Section 2.8 of [26]. For multivariate  $x$ , kernel smoothers suffer from “curse of dimensionality” and additive regression modeling is an approach to address the dimensionality issue;

see [26, pp. 82-89]. In the case of American options,  $x$  is only two-dimensional (consisting of  $S/K$  and  $t^*$ ), and therefore dimensionality is not an issue in the kernel smoothers used by Broadie et al. [22] to develop nonparametric pricing formulas. However, as pointed out above, this kernel approach assumes constant  $\sigma$  and  $r$  in the pricing formula. Replacing  $(\sigma, r)$  by  $(\hat{\sigma}_t, \hat{r}_t)$  and including it also in  $x$  will lead to difficulties due to curse of dimensionality with the kernel approach (involving 4-dimensional  $x$ ). Whereas [29] advocates the use of linear splines in additive regression modeling, we have also included quadratic splines in (12) to ensure smoothness of the price in  $z$ , as  $\partial P/\partial S$  is involved in hedging option positions that will be discussed in the next section. Besides the basic variables  $u$  and  $z$ , the additive regression model (12) also involves the nonlinear function  $w$  of  $u$  and  $z$ , which is suggested by conventional theory leading to (7) and (8), as explained above.

### C. Simulation Study of Performance

Following the simulation study in [21] on the performance of nonparametric pricing of European options on non-dividend-paying stocks, we simulated under the risk-neutral measure (2) a two-year sample of daily stock prices  $S_t = S_0 \exp(\sum_{i=1}^t X_i)$  by drawing 506 i.i.d. normal random variables  $X_i$  with mean  $\mu/253$  and variance  $\sigma^2/253$  (assuming 253 trading days per year), where  $\mu = r - \sigma^2/2$  ( $d = 0$ ). We actually initialized at  $S_{-60} = 50$  and generated  $S_{-59}, \dots, S_0$  accordingly so that we had asset prices  $S_{-60}, \dots, S_{-1}$  to compute the volatility estimate  $\hat{\sigma}_0$ . For each sample path of 506 daily stock prices, we used the rules of the Chicago Board Options Exchange (CBOE) to construct the strike prices and expiration dates of options outstanding on the stock: At any time, there are 4 expiration dates of these CBOE stock options – the current month, the next month, and the following two expirations from a quarterly schedule. The strike prices are set at multiples of \$5 in the \$25 to \$200 range. When an option expires and a new expiration date is introduced, two strike prices closest to the current stock price are chosen for the new options. There are 5620 options thus generated. The American option prices  $P$  and their hedge parameters  $\Delta = \partial P/\partial S$  were computed via (7) by using a linear spline approximation to  $\bar{z}(\cdot)$  as in [15].

Table I shows the range of stock prices and the frequency of strike prices taking the values 40, 45,  $\dots$ , 65 for different times  $t$  (during the two-year period) in the sample of 5620 options thus generated. It shows markedly different distributions of  $(S, K)$  between the first 100 and the last 106 trading days. However,  $S/K$  has tighter and more similar distributions over different time intervals. Fig. 1 (left panel) shows the difference  $\hat{P} - P$  between the estimated and actual option prices at  $253t^* = 20, 60, 120$  and  $S/K$  ranging between 0.85 and 1.2, where

$P$  is the actual price computed from (7) and  $\hat{P}$  is the price given by the additive model (12) in which  $\alpha, \alpha_j, \beta_j$  and  $\gamma_j$  are estimated by least squares from the sample of 5620 option prices. The values of  $S/K$ , at each of the three values of  $t^*$ , in the sample are represented by vertical bars along the horizontal axis. The results show that  $P$  is well approximated by  $\hat{P}$ , especially over intervals of  $S/K$  values that occur frequently in the sample. Fig. 1 (right panel) shows the corresponding differences  $\hat{\Delta} - \Delta$  between the estimated and actual hedge parameters and reveals a pattern similar to that of  $\hat{P} - P$ .

INSERT TABLE 1 AND FIGURE 1 ABOUT HERE

In Fig. 1 we have assumed that  $\sigma$  and  $r$  are known constants, as in Black-Scholes setting underlying (7). We next consider two other scenarios. In the first scenario,  $\sigma$  and  $r$  are unknown in the additive model (12) (in which  $u = -\sigma^2 t^*$  and  $\rho = r/\sigma^2$ ) and are replaced by the estimates  $\hat{\sigma}_t$  and  $\hat{r}_t$  described above. In the simulation study we assume that the yields  $\hat{r}_t$  of the short-maturity Treasury bill used to estimate  $r$  are independent and normally distributed with mean  $r$  and standard deviation 0.01. Besides the 5620 option prices as the training sample, we also generated 500 additional out-sample paths of stock prices (from the same geometric Brownian motion as before), yielding 500 American options with strike prices 45, 50 and 55 chosen at random, for each of the expiration dates of 1, 2, 3 and 6 months. Table II gives  $E(|\hat{P} - P|)$  for this scenario, where the expectation is estimated from the average values of the 500 (out-sample) options.

In the second scenario, not only are  $\sigma$  and  $r$  unknown but the actual option prices  $P_A(t_i^*, S_i)$  are the following perturbed versions of (7):

$$P_A(t_i^*, S_i) = P(t_i^*, S_i) + e^{-rt_i^*}(S_i + K_i \varepsilon_i), \quad (14)$$

where the  $\varepsilon_i$  are independent and uniformly distributed on  $[-0.1, 0.1]$  and  $i$  ranges over the 5620 in-sample and 500 out-sample options. Table III gives  $E(|P_A - \hat{P}|)$  and  $E(|P_A - P|)$  for this scenario, in which  $P$  assumes the Black-Scholes setting with known  $r$  and  $\sigma$  and is therefore given by (7).

INSERT TABLES II AND III ABOUT HERE

For European options, Hutchinson, Lo and Poggio [21] pointed out the importance of evaluating how a pricing formula performs in hedging applications. The Black-Scholes theory assumes continuous rebalancing of portfolios, for which -1 unit of the option and  $\Delta (= \partial p / \partial S)$

units of the risky asset form a portfolio which is risk-free at all times  $t$ . More generally, for a dynamic portfolio consisting of  $-1$  unit of the option and  $y(t)$  units of the risky asset, Ito's formula gives

$$\text{Var}\{df_t|(S_u, \Delta_A(u), y(u)), u \leq t\} = \sigma^2 S_t^2 \{y(t) - \Delta_A(t)\}^2 dt,$$

where  $f_t$  is the value of the portfolio at time  $t$ ,  $df_t = f_{t+dt} - f_t$ ,  $\Delta_A = \partial P_A / \partial S$ , with  $P_A$  being the option's actual price which may not be equal to  $P$  when the Black-Scholes assumptions do not hold. Hence, if we use our nonparametric pricing formula  $\hat{P}$  for hedging with American options, we can evaluate its performance by

$$\kappa_{\hat{P}} = E \left\{ \int_0^\tau (S_t/K)^2 (\Delta_A(t) - \hat{\Delta}(t))^2 dt \right\}, \quad (15)$$

where  $\tau$  is the time of exercise and  $\hat{\Delta} = \partial \hat{P} / \partial S$ . In practice, continuous rebalancing is not possible. If rebalancing is done only daily, then  $(S/K)^2 (\Delta_A - \hat{\Delta})^2$  in (15) is replaced by a step function that stays constant on intervals of width  $1/253$ . An alternative measure of performance is obtained by changing  $T$  to  $\tau$  in the measure proposed for European options in [21], leading to

$$\xi_{\hat{P}} = E \left\{ e^{-r\tau} |V_{\hat{P}}(\tau)| \right\}, \quad (16)$$

where  $V_{\hat{P}}(t)$  is the value of the replicating portfolio at time  $t$  that rebalances (according to the pricing formula  $\hat{P}$ ) between the risky asset and the riskless asset, together with  $-1$  unit of the option [21, pp.868-869]. In the Black-Scholes framework with continuous rebalancing,  $\xi_P = 0$  for the pricing formula  $P$  given by (7) with known  $r$  and  $\sigma$ . Tables II and III assume daily rebalancing and give the values of  $\kappa_{\hat{P}}$ ,  $\kappa_P$ ,  $\xi_{\hat{P}}$ ,  $\xi_P$  that are computed from the 500 simulated options. They show that the hedging performance of the nonparametric pricing formula (with unspecified  $r$  and  $\sigma$  to be estimated from the data) compares favorably with that of the parametric pricing formula under the Black-Scholes model (Table II), and is markedly superior to that of the parametric pricing formula when the actual prices are given by (14).

#### IV. BASIS FUNCTIONS IN MONTE CARLO APPROACH

As pointed out at the end of Section II, complex American options involve multiple sources of uncertainties over time, each represented by a state variable. The computational complexity of the backward induction algorithm of dynamic programming via discretization of the state space grows exponentially in the number of state variables. To circumvent the

“curse of dimensionality”, simulation-based methods that involve only representative samples of the states have been introduced. Whereas the simulation approach provides an obvious way to evaluate the expected value of the discounted reward  $e^{-r\tau} R_\tau$  for a given stopping rule  $\tau$ , it is not feasible to optimize over (uncountably many)  $\tau$ 's the simulated mean rewards without resorting to approximations. These approximations can be broadly classified into two types: reduction of the state space via aggregated or sampled states, and use of basis functions to approximate value functions.

#### *A. State Aggregation and Randomization*

State aggregation is a useful method for the numerical solution, via substantial reduction of the size of the state space, of certain types of stochastic control problems; see Section 6.6 of [30]. Barraquand and Martineau [31] refined this method to price an American option on  $m$  risky assets by using backward induction. Specifically, they partition the  $m$ -dimensional state space into  $k$  disjoint regions  $R_1, \dots, R_k$  by dividing the one-dimensional payoffs of the states into  $k$  strata. After generating a large number of sample paths of the  $m$ -dimensional asset price process, their approach is to count at time  $t$  (assumed to be discrete, taking values in  $0, \delta, \dots, N\delta$ ) the relative frequency  $\phi_{ij}(t)$  that are in  $R_i$  at time  $t$  and in  $R_j$  at time  $t + \delta$ . The payoff of region  $R_i$  if stopping occurs at time  $t$  is defined as the average of corresponding payoffs of those sample paths that fall in  $R_i$  at time  $t$ . This leads to an optimal stopping problem for the approximating Markov chain with  $k$  states, which can be conveniently solved by backward induction. Although the payoff is a natural variable to consider in partitioning the state space, using it alone may not give an adequate approximation to the optimal stopping region. Raymar and Zwecher [32] propose to incorporate an additional variable, such as the second largest or the median of the  $m$  asset prices, in the case where the payoff is the maximum of the  $m$  asset prices. While it provides some improvement for this special case, how to choose the second variable is unclear for more general situations.

Instead of state aggregation, Broadie and Glasserman [33] propose to circumvent the curse of dimensionality in standard state space discretization by using a more tractable collection of sampled states. Specifically, they use a random tree with  $b$  branches per node to approximate the vector-valued asset price process with the root node at the initial price vector. At each time  $t$  (assumed to be discrete, taking values between 1 and  $N$ ), the option value can be estimated by the maximum of the immediate exercise value and the average (over the  $b$  randomly generated branches) of the succeeding discounted optimal values, similar to the usual backward induction algorithm (such as (6)). They call this the “high estimate” as

it tends to over-estimate the actual option value. They also propose a “low estimate”, with downward bias, by separating the branches at each node into two sets, one of which is used to decide whether or not to exercise and the other to estimate the continuation value. The total number of randomly generated nodes is  $b(b^N - 1)/(b - 1)$ , which is computationally expensive when  $N$  (the number of possible exercise times) and  $b$  are large. To reduce computational cost, they use a relatively small  $b$ , say  $b = 50$ , and propose to estimate the actual option value by an interval with the low and high estimates as end-points.

### B. Neuro-Dynamic Programming and Choice of Basis Functions

Tsitsiklis and Van Roy [4], [5] consider American options that can be exercised only at a finite number of evenly spaced times, i.e., the so-called *Bermudan options*, for which the price is given by an optimal stopping problem of the form  $\sup_{\tau} E\{e^{-r\tau}g(X_{\tau})\}$  for a Markov chain  $\{X_n, 0 \leq n \leq N\}$ . For a bounded function  $f$  on the state space, define the function  $Pf$  by

$$(Pf)(x) = E\{f(X_{n+1})|X_n = x\}. \quad (17)$$

Let  $a = e^{-r}$ . Then the backward induction algorithm of dynamic programming generates the value functions  $V_N, V_{N-1}, \dots, V_0$  by

$$V_N = g, \quad V_n = \max(g, aPV_{n+1}) \text{ for } n = N - 1, \dots, 0,$$

so that the price is given by  $V_0(X_0)$ . Instead of working directly with  $V_n$ , [5] uses the “cost-to-go” functions  $Q_n$  defined by

$$Q_{N-1} = aPg, \quad Q_n = aP \max(g, Q_{n+1}) \text{ for } n = N - 2, \dots, 0, \quad (18)$$

i.e.  $Q_n = aPV_{n+1}$  so that  $V_0 = \max(g, Q_0)$ . Their approach is to compute these  $Q$ -functions by (i) simulating a large number  $I$  of trajectories  $x^{(i)} = \{x_n^{(i)}, 0 \leq n \leq N\}$ , and (ii) approximating the operator  $P$  by a projection operator onto a space of basis functions.

Unless the state space is finite or of dimension 1 or 2, direct computation of (18) via discretization of the state space is prohibitively difficult. Noting that  $P$  in (17) is the conditional expectation operator, we can represent  $Q_n/a$  as the regression function of  $\max\{g(X_{n+1}), Q_{n+1}(X_{n+1})\}$  on  $X_n$ , which is a function of the state vector  $X_n$ . A key idea of neuro-dynamic programming is to use neural networks (hence the prefix “neuro”) or other multivariable function approximation architectures to approximate the regression functions  $Q_n$ . If we use neural networks  $f_n(x) = \beta_{0,n} + \sum_{k=1}^{K_n} \beta_{k,n} h(\gamma_{k,n} + \alpha_{k,n}^T x)$  to approximate  $Q_n(x)$ ,

in which  $h$  is the logistic function, then we have to estimate the linear parameters  $\beta_{k,n}$  and nonlinear parameters  $\gamma_{k,n}$  and  $\alpha_{k,n}$ . Likewise radial basis function networks also involve nonlinear parameters. Using a linear combination  $\sum_{k=1}^K \beta_{k,n} \phi_k(x)$  of basis functions  $\phi_1(x), \dots, \phi_K(x)$  to approximate  $Q_n(x)$  has the computational advantage that  $\beta_{k,n}$  can be conveniently estimated (from a large number  $I$  of simulated trajectories) by linear least squares.

The discretization method of evaluating the value functions  $V_n$  at a prespecified grid of points fails to adapt to the underlying distribution of  $X_n$ . Although simulation-based approach also yields a discrete subset of the state space on which  $V_n$  is to be evaluated, this discrete subset is generated from the distribution of  $X_n$ . Specifically, from the  $I$  trajectories  $\{x_n^{(i)}, 0 \leq n \leq N\}$  drawn independently from the distribution of  $(X_0, X_1, \dots, X_N)$ , we can estimate  $(\beta_{1,n}, \dots, \beta_{K,n})$  by  $(\hat{\beta}_{1,n}, \dots, \hat{\beta}_{K,n})$  that minimizes

$$\sum_{i=1}^I [a \max\{g(x_{n+1}^{(i)}), \sum_{k=1}^K \hat{\beta}_{k,n+1} \phi_k(x_{n+1}^{(i)})\} - \sum_{k=1}^K b_k \phi_k(x_n^{(i)})]^2$$

over  $(b_1, \dots, b_K)$ , yielding the approximation  $\hat{Q}_n(x) := \sum_{k=1}^K \hat{\beta}_{k,n} \phi_k(x)$  to  $Q_n(x)$ . This is the approach used by Tsitsiklis and Van Roy [5]. Earlier they considered in [4] the corresponding infinite-horizon discounted optimal stopping problem for  $\{X_n, n \geq 0\}$ , whose value function  $V$  satisfies the dynamic programming equation  $V = \max(g, aPV)$ , and made use of simulation (of a single trajectory of the Markov chain) and stochastic approximation to determine the parameters  $\beta_k$  in the approximation  $\sum_{k=1}^K \beta_k \phi_k(x)$  of  $Q(x) := aPV(x)$ . Such simulation-based optimization schemes are also used in the broader context of multidimensional stochastic control problems in neuro-dynamic programming; see [34].

An important issue concerning the performance of this approach is accumulation of errors (from  $Q_{n+1}$  to  $Q_n$  for  $n = N - 2, \dots, 0$ ) and is discussed in Section IV of [5]. Instead of directly approximating  $V_0$  by  $\max(g, \hat{Q}_0)$  for which errors have been accumulated through  $N$  stages, Longstaff and Schwartz [35] propose the following alternative. First the optimal stopping rule  $\tau = \inf\{n \leq N - 1 : g(X_n) \geq Q_n(X_n)\}$  ( $\inf \emptyset = N$ ) can be approximated by using  $\hat{Q}_n(X_n)$  as a substitute for  $Q_n(X_n)$ . Using this approximation, the stopping time  $\tau_i$  can be determined for each trajectory  $x^{(i)}$ , and  $V_0$  can then be approximated by

$$\hat{V}_0 = \left\{ \sum_{i=1}^I e^{-r\tau_i} g(X_{\tau_i}) \right\} / I. \quad (19)$$

The approximation  $\sum_{k=1}^K \beta_{k,n} \phi_k$  of  $Q_n$  can be regarded as the projection of  $Q_n$  into some Hilbert space spanned by the basis functions  $\phi_1, \dots, \phi_K$ ; see [4], [5]. Since the stopping rule

$\tilde{\tau} = \inf\{n \leq N - 1 : g(X_n) \geq \sum_{k=1}^K \beta_{k,n} \phi_k(X_n)\}$  is suboptimal, (19) yields a Monte Carlo estimate of the corresponding lower bound of  $V_0$ . The tightness of this lower bound can be assessed by examining how much it differs from some suitably chosen upper bound of  $V_0$ . Haugh and Kogan [36] recently developed a simulation-based upper bound by making use of the representation

$$V_0 = \inf_M E \left\{ \max_{0 \leq n \leq N} (Z_n - M_n) \right\}, \quad (20)$$

where  $Z_n = e^{-rn}g(X_n)$  and  $\inf_M$  is taken over all martingales  $\{M_n, 0 \leq n \leq N\}$  with  $M_0 = 0$ . The minimization problem (20) can be regarded as the dual of the primal problem of maximizing  $EZ_\tau$  over all stopping rules  $\tau$ ; this was noted earlier by Rogers [37] from a general result on optimal stopping (in which time  $t$  can be continuous and unbounded and  $Z_t$  is a general right-continuous stochastic process) due to Davis and Karatzas [38].

Another important issue in the preceding approach is the choice of  $K$  and of the basis functions  $\phi_1, \dots, \phi_K$ . A related issue is whether other approximation schemes (such as neural networks) involving nonlinear parameters should be used instead, as in Haugh and Kogan [36] who use a class of neural networks (single-layered perceptions) to approximate the  $Q$ -functions. These nonlinear parameters are estimated in [36] by the Levenberg-Marquardt method (for iterative optimization under the least squares criterion) which is computationally expensive, especially for a large number  $K_n$  of hidden units and a large training sample size  $I$ . To reduce the computational burden, [36] uses  $K_n \equiv K$  and  $\widehat{\beta}_{k,n+1}$  as the starting value of the iterative algorithm to compute  $\widehat{\beta}_{k,n}$ . The choice of  $K$  is based on cross validation only at stage  $N - 1$ . Moreover,  $I$  is taken to be relatively small ( $< 5000$ ) in [36] compared with that of 50,000 to 100,000 used in [35], where linear least squares is used to estimate the parameters in the additive regression model consisting of Laguerre polynomials in the individual state variables and the cross products of these terms.

Since a very large sample of trajectories is generated to ensure the accuracy of Monte Carlo estimates, it is desirable to use the entire sample (instead of only a small fraction of it) to determine good approximations to the  $Q$ -functions. Accordingly, linearly parameterized approximations whose parameters can be estimated by linear least squares are preferable to nonlinearly parameterized approximations because of computational tractability in the presence of a large training sample and consequently also a large number of basis functions. On the other hand, the use of predetermined basis functions like Laguerre polynomials in [35] does not adapt to the data and has much less flexibility than the neural networks used in [36], where the nonlinear parameters in the basis functions can be tuned to the data. To resolve the dilemma between computational complexity and spatial adaptivity for large

data sets, Lai and Zhang [39] propose to use additive regression splines of the type used in Section IIIB, in which the knots of the splines are placed at certain percentiles of the data for spatial adaptivity. Although computation of the percentiles involves sorting which is computationally expensive for large data sets, we can use a relatively small subsample to determine the percentiles for knot placement. Another important idea in Section IIIB is the transformation of  $(t^*, S)$  into the variables  $u, z$  and  $w$  used in the additive regression model (12). This idea is extended in [39] to general state variables, leading to transformations that incorporate the important features of the problem and including in particular the payoff as one of the transformed variables, similar to what Barraquand and Martineau [31] used in their state aggregation approach. For further details, including the choice of variables and of the number  $K$  of basis functions relative to the sample size  $I$ , see [39].

### C. Theoretical Analysis and Continuous-Time Extensions

We now give some consistency and convergence rate results for the preceding approach to solving discrete-time optimal stopping problems for multivariate Markov chains via simulation and regression. Fix  $n \in \{0, 1, \dots, N-1\}$  and let  $\{(x_n^{(i)}, x_{n+1}^{(i)}), 1 \leq i \leq I\}$  be a sample of size  $I$  from the distribution of  $(X_n, X_{n+1})$ . The essence of the preceding approach lies in estimation of the regression function  $Q_n(x) = aE\{\max(g(X_{n+1}), Q_{n+1}(X_{n+1}))|X_n = x\}$ . Let  $\mathcal{X}$  denote the state space and  $\mu_n$  the distribution of  $X_n$ . Consider the Hilbert space  $\mathcal{H}$  of functions  $h$  on  $\mathcal{X}$  such that  $\|h\|^2 := \int h^2(x)d\mu_n(x) < \infty$  (corresponding to the inner product  $\langle h_1, h_2 \rangle = E\{h_1(X_n)h_2(X_n)\}$ ). Assume that  $g$  and  $Q_{n+1}$  belong to  $\mathcal{H}$ . Then  $Q_n \in \mathcal{H}$ . Let  $\{\phi_1, \phi_2, \dots\}$  be an orthonormal basis of  $\mathcal{H}$ , and let  $\mathcal{H}_K$  be the linear subspace of  $\mathcal{H}$  spanned by  $\{\phi_1, \dots, \phi_K\}$ . Let  $\psi = a \max(g, Q_{n+1})$ .

We can estimate  $Q_n$  by first approximating it by its projection  $q := \sum_{k=1}^K \theta_k \phi_k$  in  $\mathcal{H}_K$ , and then estimating  $\theta = (\theta_1, \dots, \theta_K)^T$  by the least squares estimate

$$\hat{\theta} = \left( \sum_{i=1}^I \Phi_i \Phi_i^T \right)^{-1} \sum_{i=1}^I \Phi_i \psi(x_{n+1}^{(i)}), \quad (21)$$

where  $\Phi_i = (\phi_1(x_n^{(i)}), \dots, \phi_K(x_n^{(i)}))^T$ . Let  $\hat{q} = \sum_{k=1}^K \hat{\theta}_k \phi_k$ .

*Theorem 1:* As  $I \rightarrow \infty$  and  $K \rightarrow \infty$  such that  $K/I \rightarrow 0$ ,

$$\|\hat{q} - Q_n\|^2 = \|Q_n - q\|^2 + O_p(K/I),$$

and therefore  $\|\hat{q} - Q_n\| \rightarrow 0$  in probability.

*Proof:* Since  $E\{\psi(X_{n+1})|X_n = x\} = Q_n(x) = q(x) + \tilde{q}(x)$ , where  $\tilde{q} \perp \mathcal{H}_K$ , it follows that  $\psi(x_{n+1}^{(i)}) = \Phi_i^T \theta + \tilde{q}(x_n^{(i)}) + \epsilon_i$ , with

$$E\epsilon_i^2 < \infty, E(\epsilon_i|x_n^{(i)}) = 0, \int \tilde{q}(x)\phi_k(x)d\mu_n(x) = 0 \quad \text{for } k = 1, \dots, K, \quad (22)$$

recalling that  $(x_i^{(i)}, x_{n+1}^{(i)})$  are i.i.d. and having the same distribution as  $(X_n, X_{n+1})$ . Therefore

$$\hat{\theta} = \theta + \left( \sum_{i=1}^I \Phi_i \Phi_i^T \right)^{-1} \left\{ \sum_{i=1}^I \Phi_i \tilde{q}(x_n^{(i)}) + \sum_{i=1}^I \Phi_i \epsilon_i \right\}. \quad (23)$$

With probability 1, the empirical distribution of  $\{x_n^{(i)} : 1 \leq i \leq I\}$  converges weakly to  $\mu_n$  and therefore

$$\begin{aligned} \|\hat{q} - q\|^2 &= \int (\hat{q}(x) - q(x))^2 \mu_n(x) \\ &\sim \sum_{i=1}^I \left\{ (\hat{\theta} - \theta)^T \Phi_i \right\}^2 / I = (\hat{\theta} - \theta)^T \sum_{i=1}^I \Phi_i \Phi_i^T (\hat{\theta} - \theta) / I. \end{aligned} \quad (24)$$

Since  $E\{\Phi_i \tilde{q}(x_n^{(i)})\} = 0$  and  $E(\epsilon_i|x_n^{(i)}) = 0$  by (22), and since  $E\{\phi_k^2(x_n^{(i)})\} = 1$  and  $E\{\phi_k(x_n^{(i)})\phi_j(x_n^{(i)})\} = 0$  for  $j \neq k$ , it follows from the law of large numbers and the functional central limit theorem that the asymptotic distribution of

$$\left( \sum_{i=1}^I \Phi_i \Phi_i^T \right)^{-1/2} \left( \sum_{i=1}^I \Phi_i \tilde{q}(x_n^{(i)}), \sum_{i=1}^I \Phi_i \epsilon_i \right)$$

is that of  $((\|\phi_1 \tilde{q}\| Z_1, \dots, \|\phi_K \tilde{q}\| Z_K)^T, (\|\phi_1 \sigma\| Z_{K+1}, \dots, \|\phi_K \sigma\| Z_{2K})^T)$ , where  $\sigma^2(x) = \text{Var}(\epsilon_i|x_n^{(i)} = x)$  and  $Z_1, \dots, Z_{2K}$  are i.i.d. standard normal random variables. Combining this with (23) and (24) yields

$$\|\hat{q} - q\|^2 = (1 + o_p(1)) \sum_{k=1}^K \left\{ \|\phi_k \tilde{q}\|^2 + \|\phi_k \sigma\|^2 \right\} / I = O_p(K/I).$$

Since  $\|\hat{q} - Q_n\|^2 = \|q - Q_n\|^2 + \|\hat{q} - q\|^2 + 2 \langle \hat{q} - q, q - Q_n \rangle$ , it then follows from the Cauchy-Schwarz inequality that  $\|\hat{q} - Q_n\|^2 = \|Q_n - q\|^2 + O_p(K/I)$ .

*Remark:* Instead of letting  $K \rightarrow \infty$  as in Theorem 1, Longstaff and Schwartz [35] assume  $K$  to be fixed and use the strong law of large numbers to show that  $\hat{q}(x)$  converges with probability 1 as  $I \rightarrow \infty$ , for every fixed  $x$ , and that given  $\varepsilon > 0$ , there exists  $K$  large enough such that  $P\{|Q_n(x) - \hat{q}(x)| > \varepsilon\} \rightarrow 0$  as  $I \rightarrow \infty$ . Theorem 1 establishes the stronger result that  $\|Q_n - \hat{q}\| \rightarrow 0$  in probability as  $I \rightarrow \infty$  and  $K \rightarrow \infty$  with  $K/I \rightarrow 0$ . It also decomposes

$\|Q_n - \widehat{q}\|^2$  into the squared bias  $\|Q_n - q\|^2$  plus a squared error of order  $O_p(K/I)$  for  $\widehat{\theta} - \theta$ . In practice, for  $n \leq N - 2$ ,  $\psi = a \max(g, Q_{n+1})$  has to be replaced by  $\widehat{\psi} = a \max(g, \widehat{q}_1)$  in (21) and the preceding proof can be modified in conjunction with an induction argument to show that

$$\begin{aligned} \|Q_n - \widehat{q}\|^2 &\leq \sum_{j=1}^{N-1-n} (aK)^j \|Q_{n+j} - \widehat{q}_j\|^2 + \|Q_n - q\|^2 + O_p(K/I) \\ &\leq \sum_{j=0}^{N-1-n} (aK)^j \{ \|Q_{n+j} - q_j\|^2 + O_p(K/I) \}, \end{aligned} \quad (25)$$

where  $q_0 = q$ ,  $q_j$  is the projection of  $Q_{n+j}$  into  $\mathcal{H}_K$  and  $\widehat{q}_j$  is the corresponding estimate. Note that the norm  $\|Q_{n+j} - q_j\|$  in (25) can vary with  $j$ ; it is the  $L_2$ -norm with respect to the measure  $\mu_{n+j}$  induced by the distribution of  $X_{n+j}$ .

Theorem 1 shows the trade-off between the squared bias  $\|Q_n - q\|^2$ , which decreases with the number  $K$  of basis functions, and the squared estimation error  $\|\widehat{q} - q\|^2$  arising from  $\widehat{\theta} - \theta$ , which grows linearly with  $K$ . It is, however, not clear how to choose  $K$  optimally relative to the sample size  $I$ . In fact, in the preceding Hilbert space framework, there are better alternatives (such as regularization and shrinkage) to the least squares estimates  $\widehat{\theta}$  when  $K \rightarrow \infty$  so as to take advantage of the fact that  $\lim_{k \rightarrow \infty} \beta_{k,n} = 0$ . Optimal rates of convergence have been established in the literature when additional smoothness conditions are satisfied by the regression function  $Q_n(x)$ . In this case, instead of an  $L_2(\mu_n)$ -basis such as the set of Laguerre polynomials in [35], we can choose local polynomials as basis functions to take advantage of the smoothness properties of  $Q_n$ . Again fix  $n$  as in the preceding discussion, and consider the case where  $\mathcal{X}$  is  $\mathbb{R}^d$  or some rectangular subset thereof so that  $X_n = (X_{n,1}, \dots, X_{n,d})$ . Let  $F_j$  be the distribution function of  $X_{n,j}$ , which will be assumed to be continuous. Using the transformation

$$x = (x_1, \dots, x_d) \rightarrow (F_1(x_1), \dots, F_d(x_d)), \quad (26)$$

we shall assume that  $\mathcal{X} = [0, 1]^d$ . A function  $h$  on  $\mathcal{X}$  is said to satisfy a Hölder condition with exponent  $\alpha \in (0, 1]$  if  $|h(x) - h(y)| \leq C|x - y|^\alpha$  for some  $C > 0$  and all  $x, y \in \mathcal{X}$ . Let  $m$  be a non-negative integer and let  $p = m + \alpha$ . A function  $h$  on  $\mathcal{X}$  is said to be  $p$ -smooth if all partial derivatives  $\partial^m h / \partial x_1^{m_1} \cdots \partial x_d^{m_d}$  ( $m_1 + \cdots + m_d = m$ ) satisfy a Hölder condition with exponent  $\alpha$ . Let  $\mathcal{S}$  be a collection of subsets of  $\{1, \dots, d\}$  such that if  $s \in \mathcal{S}$ , then  $r \in \mathcal{S}$  for every  $r \subset s$ . Let  $\mathcal{H}_s = \{h \in \mathcal{H} : h \text{ is constant in the coordinates not belonging to } s\}$ ; in other words, every  $h \in \mathcal{H}_s$  is a function only of the variables  $x_i$  with  $i \in s$ . In particular,  $\mathcal{H}_\emptyset$

is the class of constant functions. Let  $\mathcal{H}_s^0 = \{h \in \mathcal{H}_s : h \perp \mathcal{H}_r \text{ for all proper subset } r \text{ of } s\}$ . Following Stone [40], we shall assume the following conditions on  $Q_n(x)$  and  $X_n$ :

- 1)  $Q_n = \sum_{s \in \mathcal{S}} \varphi_s$  for some  $p$ -smooth functions  $\varphi_s \in \mathcal{H}_s^0$ , with  $p > d/2$ .
- 2) The joint density function of  $(F_1(X_{n,1}), \dots, F_d(X_{n,d}))$  with respect to Lebesgue measure is bounded away from 0 and  $\infty$ .

Let  $d^*$  be the maximum size of sets belonging to  $\mathcal{S}$ , i.e.  $d^* = \max_{s \in \mathcal{S}} \#(s)$ . Under Conditions 1) and 2),  $I^{-2p/(2p+d^*)}$  is the optimal rate of convergence of  $\|\widehat{Q} - Q_n\|^2$  for any estimator  $\widehat{Q}$  based on an i.i.d. sample of size  $I$  from the distribution of  $(X_n, \psi(X_{n+1}))$ . Moreover, additive regression with tensor products of univariate polynomial splines as basis functions can be used to construct estimators of  $Q_n$  that attain this optimal convergence rate; see [40, pp.128-129]. Specifically, take a positive integer  $J \sim I^{1/(2p+d^*)}$  and partition  $[0,1]$  into  $J$  subintervals  $[(j-1)/J, j/J]$ ,  $1 \leq j \leq J$ , recalling that  $\mathcal{X} = [0,1]^d$  after the transformation (26). Let  $\Pi$  be the linear space of splines that are polynomials of degree  $[p]$  or less on each subinterval and, in the case  $p \geq 1$ , also have continuous derivatives of order  $[p] - 1$  on  $[0,1]$ . A basis of  $\Pi$  is  $\{B_1, \dots, B_J\}$ , where  $B_j$  are the B-splines that satisfy

$$B_j \geq 0, \quad \sum_{j=1}^J B_j = 1 \text{ on } [0,1], \quad (27)$$

and  $B_j = 0$  outside an interval of length  $([p] + 1)/J$ . For  $s \in \mathcal{S}$ , let  $\mathbf{J}(s)$  denote the set of ordered  $\#(s)$ -tuples  $(j_l : l \in s)$  with  $1 \leq j_l \leq J$ , and define for  $\mathbf{j} \in \mathbf{J}(s)$  the tensor product

$$B_{\mathbf{s}\mathbf{j}}(x_1, \dots, x_d) = \prod_{l \in s} B_{j_l}(x_l).$$

The functions  $B_{\mathbf{s}\mathbf{j}}$  ( $\mathbf{j} \in \mathbf{J}(s)$ ), which are non-negative and sum to 1 over  $\mathbf{J}(s)$ , form a basis of the linear space  $\mathcal{F}_s$  spanned by functions  $f$  of the form  $f(x_1, \dots, x_d) = \prod_{l \in s} g_l(x_l)$  with  $g_l \in \Pi$ . Let  $\mathcal{F} = \{\sum_{s \in \mathcal{S}} f_s : f_s \in \mathcal{F}_s\}$ . Let  $\widehat{f}$  be the minimizer of  $\sum_{i=1}^I \{\psi(x_{n+1}^{(i)}) - f(x_n^{(i)})\}^2$  over  $f \in \mathcal{F}$ . In view of the basis functions  $B_{\mathbf{s}\mathbf{j}}$  of  $\mathcal{F}$ , we can compute  $\widehat{f}$  by using the least squares estimates  $\widehat{\beta}_{\mathbf{s}\mathbf{j}}$  of the coefficients of  $B_{\mathbf{s}\mathbf{j}}$  in an additive regression model with  $B_{\mathbf{s}\mathbf{j}}$ ,  $\mathbf{j} \in \mathbf{J}(s)$  and  $s \in \mathcal{S}$ , as the basis functions. Stone [40, Corollary 2.1] has shown that  $\widehat{f}$  indeed attains the optimal rate of convergence, i.e.,  $\|Q_n - \widehat{f}\|^2 = O_p(I^{-2p/(2p+d^*)})$ .

Unfortunately, this result is not directly applicable to our setting for  $n \leq N - 2$ , since  $\psi = a \max(g, Q_{n+1})$  is an unknown function, in which  $Q_{n+1}$  has also to be estimated from a training sample of size  $I$ . However, we can modify the arguments in [40] to estimate  $Q_n$

at the optimal rate. First for  $n = N - 1$ ,  $Q_n(x) = aE\{g(x_N^{(i)})|x_{N-1}^{(i)} = x\}$  can indeed be estimated by the preceding procedure at the optimal rate. We can then proceed inductively as follows. For  $n \leq N - 2$ , apply the procedure, with  $\psi$  replaced by  $\hat{\psi} = a \max(g, \hat{Q}_{n+1})$ , to estimate  $Q_n$ , where  $\hat{Q}_{n+1}$  denotes the corresponding estimate of  $Q_{n+1}$ . This is, therefore, the same idea as that in the Remark following Theorem 1. However, unlike Theorem 1 which uses an orthonormal  $L_2(\mu_n)$ -basis for which the bound in (25) grows exponentially in the number  $K$  of basis functions as  $n$  moves backwards from  $N - 1$ , the spline basis used in  $\hat{Q}_n$  has certain important features that circumvent this difficulty, leading to the following convergence rate for  $\hat{Q}_n$ .

*Theorem 2:* Suppose Conditions 1) and 2) hold for every  $n \in \{0, 1, \dots, N - 1\}$ . Then for the  $\hat{Q}_n$  defined in the preceding paragraph with  $J \sim I^{1/(2p+d^*)}$ ,

$$\|\hat{Q}_n - Q_n\|^2 = O_p(I^{-2p/(2p+d^*)}) \text{ as } I \rightarrow \infty, \text{ for } n = 0, 1, \dots, N - 1.$$

*Proof:* A key property of B-splines used in [40] is (27). For the tensor products  $B_{\mathbf{s}\mathbf{j}}$ , we also have  $\sum_{\mathbf{j} \in \mathbf{J}(s)} B_{\mathbf{s}\mathbf{j}} = 1$ . Hence in particular,  $\sum_{\mathbf{j}, \mathbf{j}' \in \mathbf{J}(s)} B_{\mathbf{s}\mathbf{j}} B_{\mathbf{s}\mathbf{j}'} = 1$ , yielding

$$\text{Var}\left(\sum_{\mathbf{j} \in \mathbf{J}(s)} \hat{\beta}_{\mathbf{s}\mathbf{j}} B_{\mathbf{s}\mathbf{j}}(x) | x_n^{(1)}, \dots, x_n^{(I)}\right) \leq \max_{\mathbf{j} \in \mathbf{J}(s)} \text{Var}(\hat{\beta}_{\mathbf{s}\mathbf{j}} | x_n^{(1)}, \dots, x_n^{(I)}) \quad (28)$$

in the proof of Theorem 3.2 of [40]. Let  $\beta_{\mathbf{s}\mathbf{j}}^*$  denote the least squares estimate when  $\psi(x_{n+1}^{(i)})$  in  $\hat{\beta}_{\mathbf{s}\mathbf{j}}$  is replaced by  $\hat{\psi}(x_{n+1}^{(i)})$ . We can make use of the arguments in Lemma 3.12 of [40] together with the induction hypothesis on  $\hat{Q}_{n+1} - Q_{n+1}$  to analyze  $E\{(\hat{\beta}_{\mathbf{s}\mathbf{j}} - \beta_{\mathbf{s}\mathbf{j}}^*)^2 | x_n^{(1)}, \dots, x_n^{(I)}\}$ . This in turn leads, via an argument like (28), to an upper bound on  $\sup_{x \in \mathcal{X}} E[\{\sum_{\mathbf{j} \in \mathbf{J}(s)} (\hat{\beta}_{\mathbf{s}\mathbf{j}} - \beta_{\mathbf{s}\mathbf{j}}^*) B_{\mathbf{s}\mathbf{j}}(x)\}^2 | x_n^{(1)}, \dots, x_n^{(I)}]$  like that in Theorem 3.2 of [40]. Note that the  $d$  in [40] corresponds to our  $d^*$  here. Combining this result with Corollary 2.1 of [40] completes the proof of Theorem 2.

The transformation (26) assumes that the distribution function  $F_k$  of  $X_{n,k}$  is known. In practice,  $F_k$  is unknown and we can replace it by the corresponding empirical distribution function of the sample. This is tantamount to placing knots of the splines at  $100j/J$ -th percentiles of the observed data, as we have done in Section IIIB. Moreover, as in [40], we need not assume that  $Q_n$  actually belongs to  $\mathcal{H}^* := \{\sum_{s \in \mathcal{S}} h_s : h_s \in \mathcal{H}_s\}$ . The convergence theory is still applicable to the problem of estimating the projection  $h$  of  $Q_n$  into  $\mathcal{H}^*$ , which is the closest approximation to  $Q_n$  (according to the norm  $\|\cdot\|$ ) in  $\mathcal{H}^*$ . In such approximation contexts, we can also use other variables  $u_1^{(i)}, \dots, u_m^{(i)}$  that are derived from the original vector  $x_n^{(i)}$  and work with an additive spline regression model of  $\psi(x_{n+1}^{(i)})$  on  $(u_1^{(i)}, \dots, u_m^{(i)})$ , as in Section IIIB and [39].

Theorem 2 assume  $N$  to be fixed while  $I \rightarrow \infty$ . When  $N$  is also large, the accumulation of errors as one estimates  $Q_{N-1}, Q_{N-2}, \dots$  recursively (backwards) becomes an important issue, especially when this method is applied to compute the value of a Bermudan option which is given by  $\max(g, Q_0)$ . An alternative approach is to use the computed  $Q_n$  to approximate the stopping rule so that the option price can be computed via (19), and it can also be extended to American options for which the stopping rule is a continuous variable. Lai and Zhang [39] use a discrete set of suitably chosen times and interpolation between these times to approximate the continuous-time stopping region, in conjunction with the preceding method to compute the  $Q$ -functions at the selected discrete set of times. As noted in Section II, one only needs a crude approximation of the optimal stopping boundary to be able to compute an American option price with reasonable accuracy. Once this stopping region is determined, it is straightforward to use simulations (or numerical integration for the simpler situations in Section II) to compute the price of an American option.

## V. CONCLUSION

Neural and other learning networks and simulation-based dynamic programming have a burgeoning literature in stochastic control. In this paper we apply these ideas to the valuation of both actively traded and over-the-counter American options. We point out several basic issues that have come up in the course of applying these ideas. One is the choice of basis functions. Closely related is the balance between statistical efficiency and computational complexity in the presence of large data sets. Another issue is the accumulation of errors in the backward induction algorithm to compute  $Q$ -functions via regression and simulation. Although American option valuation only involves the simplest kind of stochastic control problems, namely, optimal stopping, it already encounters the long-standing "curse-of-dimensionality", and how it handles this "curse" should provide useful clues for the solution of more complicated stochastic control problems.

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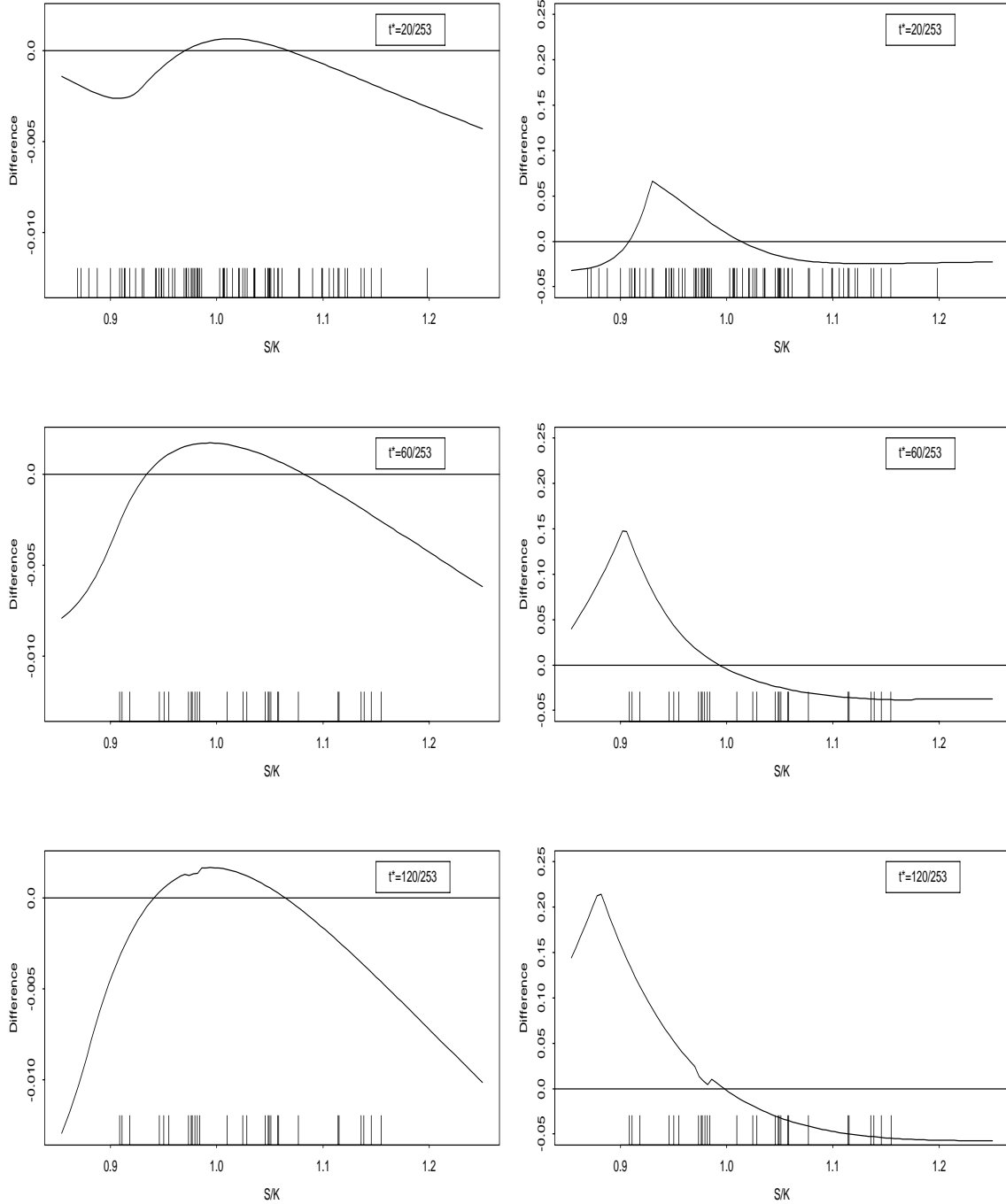
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TABLE I  
TRAINING SAMPLE OF 5620 AMERICAN OPTIONS

Time (in days)	Stock Price $S$		Number of Strik Prices $K$							
	Min	Max	$K = 55$	60	65	70	75	80	85	90
1 – 100	60.82	70.26	0	388	400	240	24	0	0	0
101 – 200	59.39	69.01	108	264	400	256	80	0	0	0
201 – 300	64.57	72.42	0	92	400	400	188	0	0	0
301 – 400	64.14	86.76	0	140	140	220	208	224	176	76
401 – 506	71.14	82.31	0	0	0	192	420	364	220	0



**Figure 1:** Differences between estimated and true prices (left) and deltas (right).

TABLE II  
 PERFORMANCE OF NONPARAMETRIC PRICING  
 FORMULA WHEN  $r$  AND  $\sigma$  ARE ESTIMATED.

$t^*$	$E \hat{P} - P $	$\kappa_{\hat{P}}$	$\kappa_P$	$\xi_{\hat{P}}$	$\xi_P$
1/12	0.0001	0.4722	0.0031	0.3138	0.1990
2/12	0.0004	0.3952	0.0080	0.2272	0.1262
3/12	0.0002	0.4590	0.0167	0.2736	0.1591
6/12	0.0002	0.6145	0.0523	0.3629	0.1959

TABLE III  
 COMPARISON OF NONPARAMETRIC AND  
 PARAMETRIC PRICING IN MODEL (14)

$t^*$	$E P_A - \hat{P} $	$E P_A - P $	$\kappa_{\hat{P}}$	$\kappa_P$	$\xi_{\hat{P}}$	$\xi_P$
1/12	0.0066	0.0999	5.8450	20.4856	3.3850	3.8244
2/12	0.0003	0.0996	6.2724	40.0066	3.3680	4.4790
3/12	0.0034	0.0974	6.9512	60.6776	3.5224	5.1614
6/12	0.0092	0.0950	11.0541	124.6119	3.8083	6.8688