# Physics-biophysics 1

## Flow of fluids

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# Contents

# What is a fluid?

- *'something that can flow'*
- *'something that has no defined shape'*
- **fluid:** a substance that cannot support shear stress
- deformation:
  - *tensile stress (tension)*
    - *'tends to change the dimensions but not the shape'*
    - force $\perp$ surface
    - $p = \frac{F}{A}$
  - *shear stress*
    - *'tends to change the shape but not the dimensions'*
    - force $\parallel$ surface
    - $\tau = \frac{F}{A}$
- *fluid* is a common name for *liquids* and *gases* — in most cases, they can be treated the same way

# Classification of fluids

1. **Compressibility**
   - A. **compressible:** its density ($\rho$) can change (eg, gases)
   - B. **incompressible:** its density ($\rho$) is constant (eg, liquids)
2. **Internal friction (viscosity)**
   - A. **viscous:** there IS internal friction
   - B. **non-viscous:** internal friction is negligible

- **ideal fluid:** incompressible and non-viscous

# Basic quantities

- **current:** the mass or volume flowing through a cross-section of a tube in unit time

$$I := \frac{\mathrm{d}m}{\mathrm{d}t} \quad \text{or} \quad \frac{\mathrm{d}V}{\mathrm{d}t},$$

where $m$ denotes the mass, $t$ is the time and $V$ stands for the volume
- the greater the cross-section area, the greater the current $\rightarrow$
- **current density:** the current flowing through a unit cross-section area

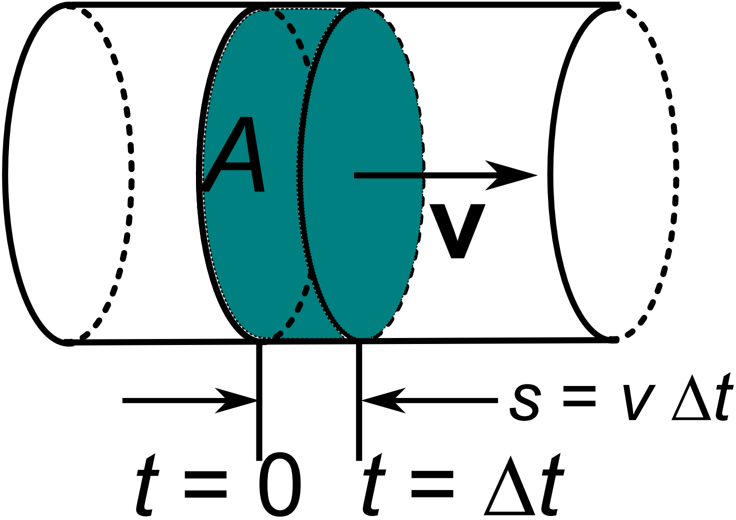
$$J := \frac{\mathrm{d}I}{\mathrm{d}A},$$

where $A$ denotes the cross-section area

# Flow of incompressible fluids

- liquids: generally incompressible
- gases: generally compressible
- even for gases, if the flow speed is not too high ($< 50\,\mathrm{m/s}$), $\rho$ does not change significantly and the *flow* can be considered incompressible
- time dependence of the flow
  - **stationary (steady):** speed and current are *independent of* time
  - **non-stationary (unsteady):** speed and current *do depend on* time

# Current and flow speed



# Current and flow speed

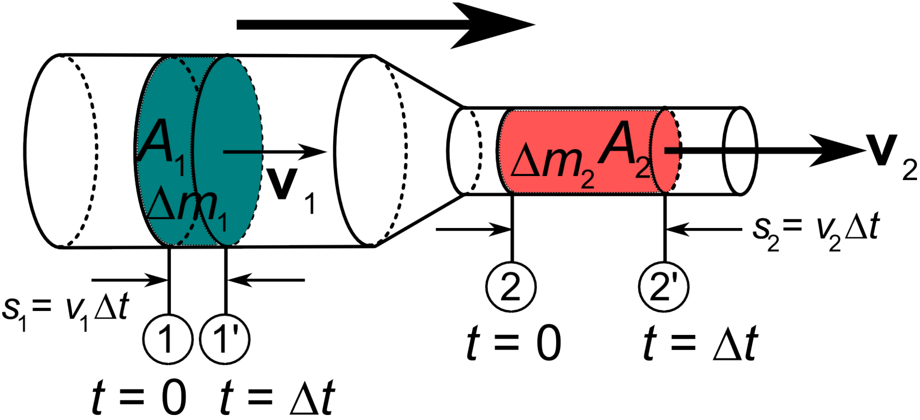
- let us assume that the fluid particles move parallel to each other with the same flow velocity $\mathbf{v}$
- this way, in time $\Delta t$ they travel a distance $s = v\Delta t$
- from the perspective of fluid flow, this means that volume $\Delta V$ has been carried through a given cross-section area $A$ of the tube, where

$$\Delta V = As = Av\Delta t$$

- thus the volume current of the flow is

$$I = \frac{\Delta V}{\Delta t} = \frac{Av\Delta t}{\Delta t} = Av$$

# Illustration



# Notations

- (1) and (2): selected discs in the fluid at time $t = 0$
- (1') and (2'): locations of the selected discs in at time $t = \Delta t$
- $\mathbf{v}_1$: the velocity of the fluid at (1) and (1')
- $\mathbf{v}_2$: the velocity of the fluid at (2) and (2')
- $A_1$: area of the cross-section at (1) and (1')
- $A_2$: area of the cross-section at (2) and (2')
- $s_1$: distance travelled by the fluid between locations (1) and (1')
- $s_2$: distance travelled by the fluid between locations (2) and (2')

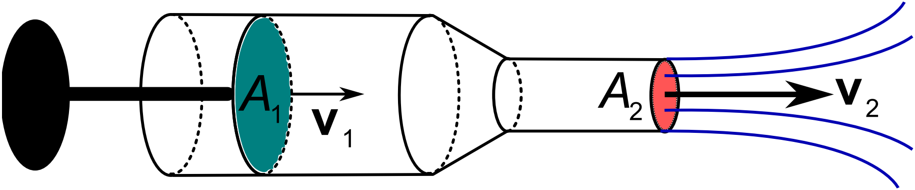
# Equation of continuity

- the fluid is incompressible, so
  - the density is the same between (1) and (1') and (2) and (2'): $\rho_1 = \rho_2 = \rho$
  - the mass flowing in at (1) is equal to the mass flowing out at (2): $\Delta m_1 = \Delta m_2 = \Delta m$
  - $\Delta m = \rho \Delta V_1 = \rho \Delta V_2 \rightarrow$
  - the volume flowing in at (1) is equal to the volume flowing out at (2): $\Delta V_1 = \Delta V_2 = \Delta V$
- $\Delta V_1 = A_1 s_1 = A_1 v_1 \Delta t$
- $\Delta V_2 = A_2 s_2 = A_2 v_2 \Delta t$
- $\Delta V_1 = \Delta V_2 \Rightarrow A_1 v_1 \Delta t = A_2 v_2 \Delta t$

$$A_1 v_1 = A_2 v_2$$

- since (1) and (2) were chosen arbitrarily, this must hold to any two cross-sections along the flow: $A \cdot v = \text{const}$
- this is the **equation of continuity**
- since $Av = I$, this means that the **volume current is constant along the tube**

# Example: a syringe




$$A_1 v_1 = A_2 v_2$$

$$A_1 \gg A_2 \Rightarrow v_2 \gg v_1$$

# Example: flow speed in blood vessels

| **Blood vessel** | **cross-section area [cm$^2$]** | **velocity [cm/s]** |
|---|---|---|
| Aorta | 4.5 | 40 |
| Arteries | 20 | 9 |
| Arterioles | 400 | 0.45 |
| Capillaries | 4500 | 0.04 |
| Veins | 40 | 4.5 |
| *Vena cava* | 18 | 10 |

Homework: what is common to all the rows?

# Flow of ideal fluids

- ideal fluid: incompressible and **non-viscous**
- no internal friction $\rightarrow$ no loss of mechanical energy
- mechanical energy: potential energy + kinetic energy

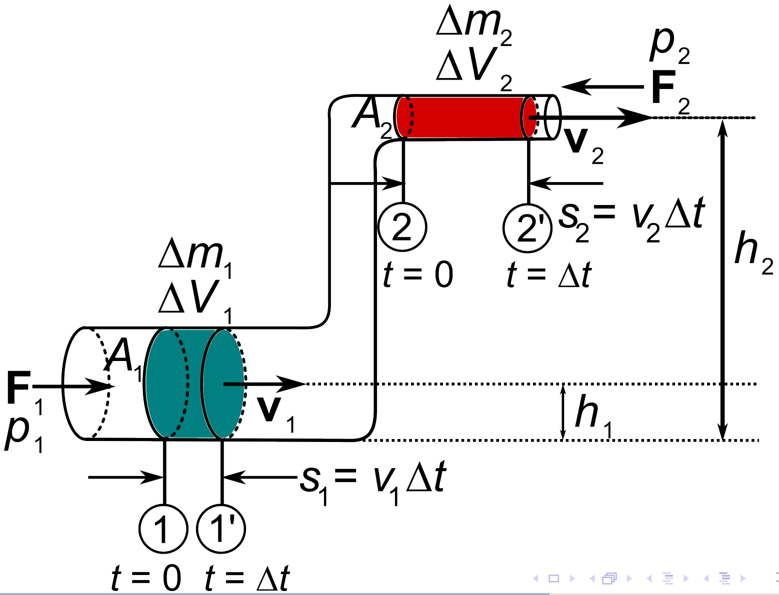
$$E = E_\text{P} + E_\text{K}$$

- the *conservation of mechanical energy* applies: external work done on the system = change in the mechanical energy of the system

$$W = \Delta E = \Delta E_\text{P} + \Delta E_\text{K}$$

- Bernoulli's law: a special form of the conservation of mechanical energy for the flow of ideal fluids

# Illustration

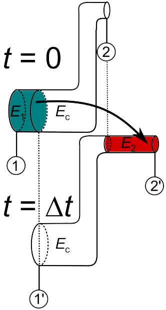


# Notations

- (1) and (2): selected discs in the fluid at time $t = 0$
- (1') and (2'): locations of the selected discs in at time $t = \Delta t$
- $\mathbf{v}_i$: the velocity of the fluid at ($i$) and ($i$') ($i \in \{1,2\}$)
- $A_i$: area of the cross-section at ($i$) and ($i$') ($i \in \{1,2\}$)
- $s_i$: distance travelled by the fluid between locations ($i$) and ($i$') ($i \in \{1,2\}$)
- $h_i$: height of the centre of mass of the fluid at ($i$) and ($i$') (as compared to an arbitrary reference level, $i \in \{1,2\}$)
- $p_i$: pressure exerted by the rest of the fluid at ($i$) and ($i$') ($i \in \{1,2\}$)
- $\mathbf{F}_i$: force exerted by the rest of the fluid at ($i$) and ($i$') ($i \in \{1,2\}$)
- $\Delta m_i$: mass of the fluid between locations ($i$) and ($i$') ($i \in \{1,2\}$)
- $\Delta V_i$: volume of the fluid between locations ($i$) and ($i$') ($i \in \{1,2\}$)
- $\rho_i$: density of the fluid between locations ($i$) and ($i$') ($i \in \{1,2\}$)

# Energy changes

- how does the energy of the fluid between (1) and (2) change?
- kinetic and potential energies only depend on the position
- $E_C$: the energy of the part of the fluid which does not change between $t = 0$ and $t = \Delta t$
- $E = E_1 + E_C$
- $E' = E_C + E_2$
- $\Delta E = E' - E = E_2 - E_1$
- only the energy of the marked sections changes



# Implications of the equation of continuity

$$\rho_1 = \rho_2 = \rho$$

$$\Delta m_1 = \Delta m_2 = \Delta m$$

$$\Delta V_1 = \frac{\Delta m_1}{\rho} = \frac{\Delta m_2}{\rho} = \Delta V_2$$

$$\Delta V_1 = A_1 s_1 = A_1 v_1 \Delta t$$

$$\Delta V_2 = A_1 s_1 = A_2 v_2 \Delta t$$

Difference between the fluid states at $t = 0$ and $t = \Delta t$: mass $\Delta m$ is transported from (1) to (2)

## What happens to this mass?

1. the rest of the fluid does work on it
2. moved from $h_1$ to $h_2$ $\rightarrow$ its potential energy changes
3. its velocity changes $\rightarrow$ its kinetic energy changes

# Work done by the rest of the fluid

- definition of pressure:

$$p = \frac{F}{A}$$

$$F = pA$$

- work done by the fluid at (1) and (2):

$$W_1 = F_1 s_1 = F_1 v_1 \Delta t = p_1 A_1 v_1 \Delta t = p_1 \Delta V = p_1 \frac{\Delta m}{\rho}$$

$$W_2 = -F_2 s_2 = -F_2 v_2 \Delta t = -p_2 A_2 v_2 \Delta t = -p_2 \Delta V = -p_2 \frac{\Delta m}{\rho}$$

- negative sign in $W_2$: the direction of the force is opposite to that of the displacement
- total external work:

$$W = W_1 + W_2 = \frac{\Delta m}{\rho} (p_1 - p_2)$$

# Potential and kinetic energies

## Gravitational potential energy: $E_\mathrm{P} = mgh$

- at (1): $E_{\mathrm{P},1} = \Delta m g h_1$
- at (2): $E_{\mathrm{P},2} = \Delta m g h_2$
- change: $\Delta E_\mathrm{P} = E_{\mathrm{P},2} - E_{\mathrm{P},1} = \Delta m g (h_2 - h_1)$

## Kinetic energy: $E_\mathrm{K} = \frac{1}{2} m v^2$

- at (1): $E_{\mathrm{K},1} = \frac{1}{2} \Delta m v_1^2$
- at (2): $E_{\mathrm{K},2} = \frac{1}{2} \Delta m v_2^2$
- change: $\Delta E_\mathrm{K} = E_{\mathrm{K},2} - E_{\mathrm{K},1} = \frac{1}{2} \Delta m (v_2^2 - v_1^2)$

# Bernoulli's equation

- conservation of mechanical energy: $W = \Delta E = \Delta E_\mathrm{P} + \Delta E_\mathrm{K}$

$$\frac{\Delta m}{\rho}(p_1 - p_2) = \Delta m g(h_2 - h_1) + \frac{1}{2}\Delta m(v_2^2 - v_1^2) \qquad // \cdot \frac{\rho}{\Delta m}$$

$$p_1 - p_2 = \rho g h_2 - \rho g h_1 + \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2$$

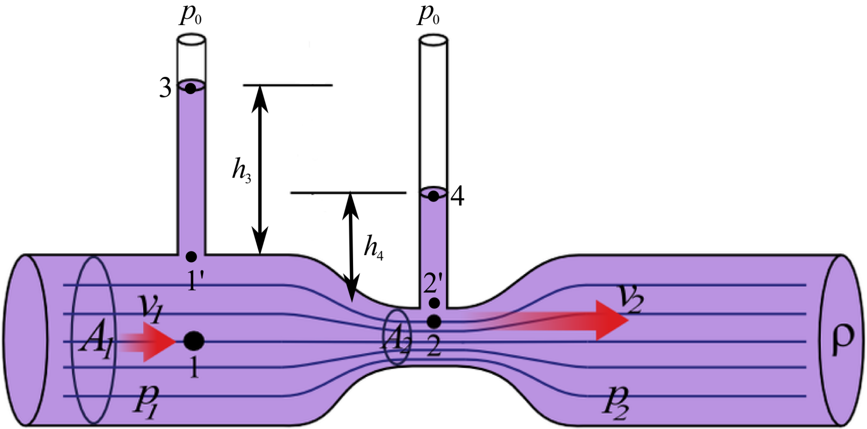
$$p_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

- fluid discs (1) and (2) were chosen arbitrarily, so this must hold to any two cross-sections along the flow: $p + \rho g h + \frac{1}{2}\rho v^2 = \mathrm{const}$
- this is **Bernoulli's law**

# Interpreting Bernoulli's law

- $p$: **static pressure**
- $\rho g h$: **hydrostatic pressure**
- $\frac{1}{2}\rho v^2$: **dynamic pressure**
- **total pressure** = static pressure + hydrostatic pressure + dynamic pressure
- Bernoulli's law in other words: *the total pressure is constant along the tube*

# Example: Venturi tube



# Example: Venturi tube

- from the equation of continuity:

$$A_1 v_1 = A_2 v_2$$

$$A_1 \gg A_2 \Rightarrow v_2 \gg v_1$$

- apply Bernoulli's law to compare 1 and 2:

$$p_1 + 0 + \frac{1}{2}\rho v_1^2 = p_2 + 0 + \frac{1}{2}\rho v_2^2$$

- since $v_2 \gg v_1 \Rightarrow p_1 \gg p_2$
- 1 and 3 cannot be compared, because they are in different tubes; but 1' and 3 can (1' is at the beginning of the vertical tube)
- the static pressures at 1 and 1' are the same: $p_{1'} = p_1$
- the dynamic pressure at 1' and 3 is 0, because the fluid does not flow in the vertical tubes
- apply Bernoulli's law to compare 1' and 3 (the reference level is now at 1'):

$$p_1 + 0 + 0 = p_0 + \rho h_3 g + 0,$$

where $p_0$ is the atmospheric pressure

# Example: Venturi tube

- 2 and 4 cannot be compared, because they are in different tubes; but 2' and 4 can (2' is at the beginning of the vertical tube)
- the static pressures at 2 and 2' are the same: $p_{2'} = p_2$
- the dynamic pressure at 2' and 4 is 0, because the fluid does not flow in the vertical tubes
- apply Bernoulli's law to compare 2' and 4 (the reference level is now at 2'):

$$p_2 + 0 + 0 = p_0 + \rho h_4 g + 0$$

- if we compare $h_3$ and $h_4$

$$h_3 = \frac{p_1 - p_0}{\rho g} \gg h_4 = \frac{p_2 - p_0}{\rho g},$$

because we have seen that $p_1 \gg p_2$

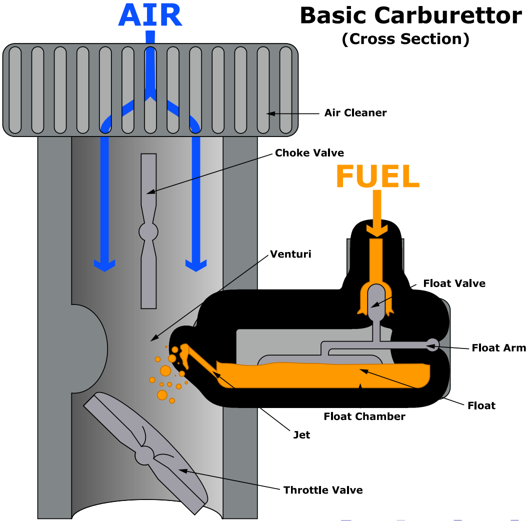
- the Venturi tube proves Bernoulli's law — *at the wider section of the tube, where flow speed is smaller, the static pressure is higher as compared to narrower sections of the tube*
- the role of vertical tubes: to make the differences in static pressure visible

# Example: Bunsen burner

- flow speed in the narrower section is greater ⇒ lower static pressure
- static pressure within the tube is less than the atmospheric pressure
- as a result, air flows into the tube
- the air influx feeds the flame at the top

# Example: carburettor

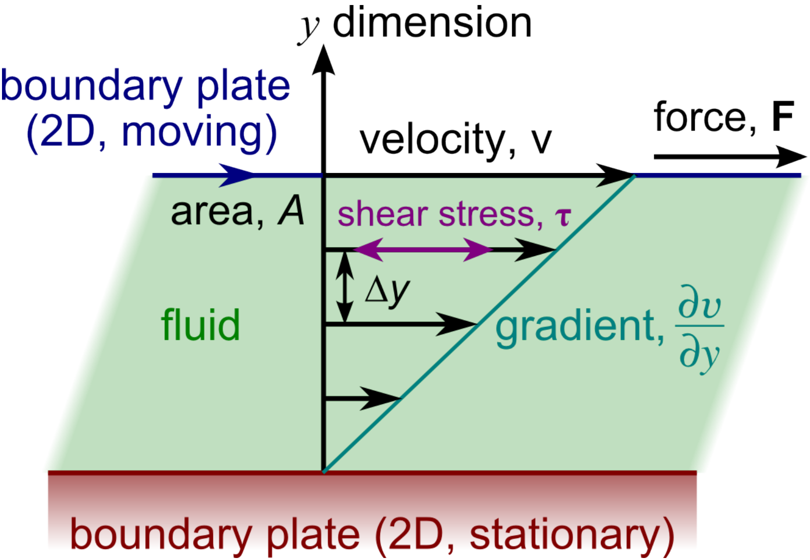
- air injected into the carburettor chamber
- in the narrow throat, the air is moving at its fastest speed and therefore it is at its lowest pressure
- low pressure in the chamber pumps the fuel into the chamber, where it is mixed with air



# What is viscosity?

- fluids (by definition) cannot *support* shearing stress
- but they *do resist* shearing motion
- interactions between the particles of the fluid ⇒ friction
- force is needed to make fluid layers slide upon each other
- viscosity: the resistance of a fluid to shearing motion due to internal friction
- everyday term for viscosity: 'thickness' (riddle: which English saying involves viscosity?)
- etymology: < Latin *viscum* 'mistletoe' (mistletoe glue was used to catch birds)

# Illustration



# What happens in a viscous fluid?

- two parallel solid plates with surface area $A$ with viscous fluid between them
- lower plate: fixed
- upper plate: pulled sideways with force $\mathbf{F}$
- bottom layers of fluid: stick to the lower plate due to adhesion, velocity: $v = 0$
- top layers of fluid: stick to the upper plate due to adhesion, velocity: $v \neq 0$
- we imagine fluid flow as fluid layers sliding upon each other
- particles within the same layer move together with the same velocity: $\mathbf{v}(y)$

# What happens in a viscous fluid?

- thickness of layers: $\Delta y$
- speed difference between neighbouring layers: $\Delta v$
- instead of the force, it is more practical to use the shear stress $\tau$

$$\tau := \frac{F}{A}$$

- how much the velocity changes across layers — *velocity gradient:*

$$\frac{\Delta v}{\Delta y}$$

- in reality, there are no homogeneous layers, the velocity changes continuously across the cross-section of the tube
- $\Rightarrow$ layer thickness $\Delta y$ should be made infinitely small
- thus the velocity gradient (also called the **shear rate**):

$$\lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} = \frac{\mathrm{d}v}{\mathrm{d}y}$$

# Newton's law of friction

- the stronger the internal friction, the more the layers move together $\Rightarrow$ the less the velocity difference between layers (shear rate)
- non-viscous fluid (eg, water): we can slide a solid plate on top of it without making the whole bulk of fluid moving
- viscous fluid (eg, honey): sliding a solid plate on its surface makes the whole bulk of fluid move
- in many fluids, shear rate $\frac{\mathrm{d}v}{\mathrm{d}y}$ is proportional to the shear stress $\tau$
- the constant of proportionality is called the **viscosity** $\eta$
- **Newton's law of friction:**

$$\tau = \frac{F}{A} = \eta \frac{\mathrm{d}v}{\mathrm{d}y}$$

- **Newtonian fluids:** fluids which obey Newton's law of friction

# Viscosity

- using Newton's law of friction, we can define viscosity:

$$\eta := \tau \left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^{-1}$$

- the unit of viscosity:

$$[\eta] = 1\,\mathrm{Pa}\cdot\mathrm{s} \qquad \left(\Leftarrow 1\,\frac{\mathrm{N}}{\mathrm{m}^2} \div \frac{\mathrm{m/s}}{\mathrm{m}} = 1\,\frac{\mathrm{Pa}}{1/\mathrm{s}}\right)$$

| **Fluid** | glycerine | blood | water | air |
|---|---|---|---|---|
| **Viscosity [Pa · s]** | 0.83 | 0.02–0.04 | 0.001 | 0.00001 |

Table 1: Typical viscosity values

# Viscosity as a diagnostic tool

- blood viscosity measurement has a great potential as a diagnostic tool
- viscosity changes in conditions of
  - myocardial infarction
  - coronary occlusion
  - arteriosclerosis
  - diabetes, *&c*
- it is not yet known whether viscosity changes are just symptoms or themselves contribute to the disorders they accompany

# Molecular origin of viscosity in gases

- in ideal gases, molecules only interact through collisions
- source of friction: *momentum exchange between layers*
- molecules in gas flow:
  - ordered motion — due to pressure difference
  - disordered motion — due to temperature (usually with much greater speed than ordered motion)
- disordered motion ⇒ molecules may enter other layers, and through collisions, change the momentum of the layer
- momentum: $\mathbf{p} := m\mathbf{v}$; Newton's 2nd law: $\mathbf{F} = m\mathbf{a} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$
- momentum exchange ⇒ force of friction between layers
- force of friction ⇒ viscosity

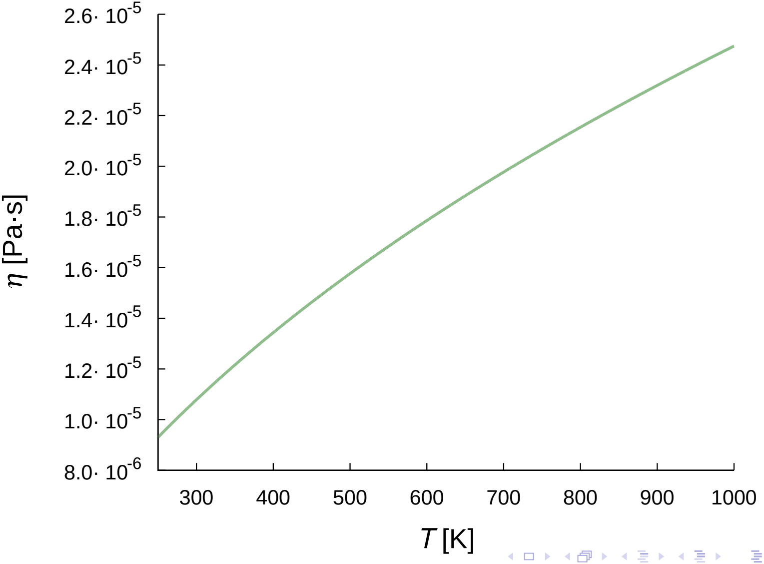
# Temperature dependence of viscosity in gases

- equipartition theory for ideal gases: $\frac{1}{2}mv^2 = \frac{f}{2}kT$, where $v$ is the average speed of gas molecules, $k$ is Boltzmann's constant, $T$ is the thermodynamic temperature (in Kelvins) and $f$ denotes the degrees of freedom
- higher temperature ⇒ faster disordered motion
- faster disordered motion ⇒ more collisions in unit time ⇒ stronger force of friction
- **the viscosity of gases increases with temperature**

$$\eta = \eta_0 \sqrt{\frac{T}{T_0}} \cdot \frac{1 + \frac{C}{T_0}}{1 + \frac{C}{T}},$$

where $\eta_0$ is the viscosity of the gas at absolute temperature $T_0$ and $C$ is called the Sutherland constant

# Temperature dependence of viscosity in gases



# Molecular origin of viscosity in liquids

- in liquids, there are strong interactions between molecules
- attractive forces ⇒ internal friction
- flow requires free space in order for the molecules to move and change places
- *How can liquids flow at all when there is not enough free space within them?*

## Frenkel's theory

- thermal motion results in occasional molecular gaps in the liquid
- it provides room for liquid molecules to move ⇒ fluid flow
- the higher the concentration of the gaps, the more easily liquid molecules can move ⇒ lower viscosity

# Temperature dependence of viscosity in liquids

- liquid molecules possess potential energy due to interactions with neighbouring molecules
- to create a gap, a molecule must be freed $\Rightarrow$ some activation energy $E$ must be invested
- source of activation energy: kinetic energy of liquid molecules
- *How many molecules have kinetic energy greater than or equal to $E$?*
- Boltzmann distribution $\Rightarrow$

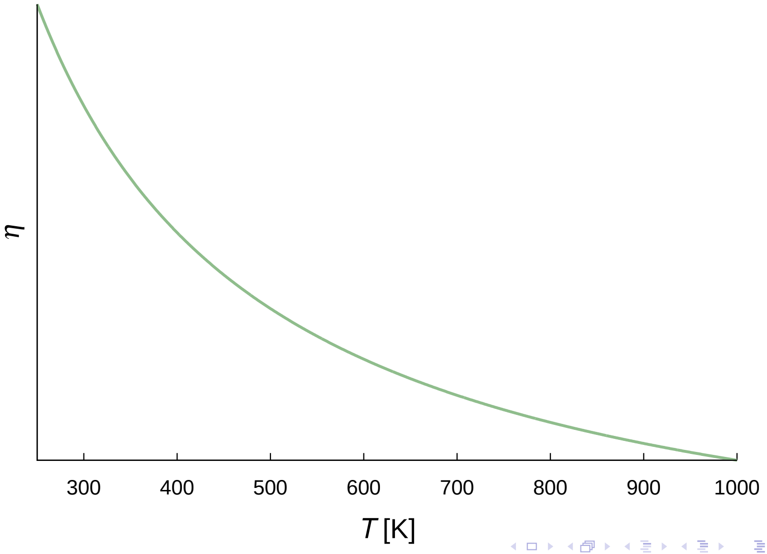
$$N \propto T \cdot e^{-\frac{E}{kT}},$$

  where $N$ is the number of molecules above the activation energy $E$, $k$ is the Boltzmann constant and $T$ is the absolute temperature
- the higher the concentration of the gaps, the lower the viscosity $\Rightarrow \eta \propto \frac{1}{N}$:

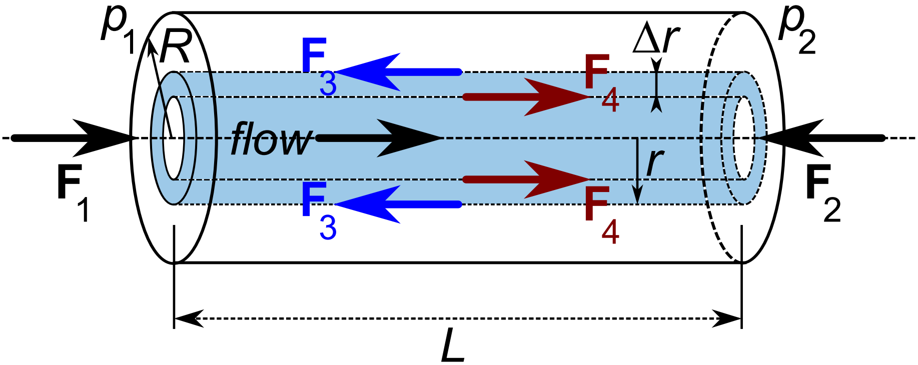
$$\eta \propto \frac{1}{T} e^{\frac{E}{kT}}$$

- **the viscosity of liquids decreases with temperature**

# Temperature dependence of viscosity in liquids



# Hagen–Poiseuille law: illustration



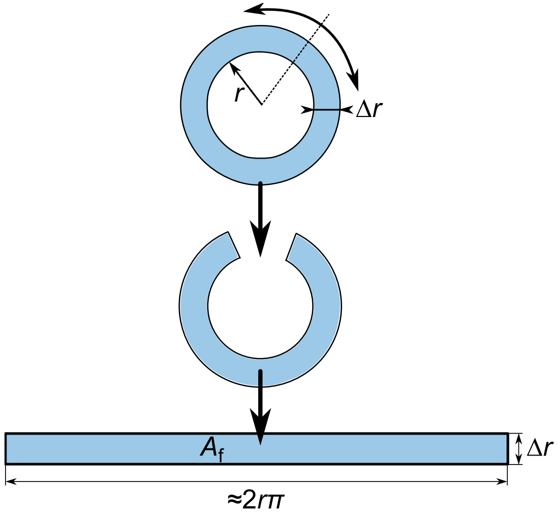
# Notations

- $p_1$: pressure exerted by the rest of the fluid at the beginning of the tube
- $p_2$: pressure exerted by the rest of the fluid at the end of the tube
- $\mathbf{F}_1$: propelling force due to fluid pressure at the beginning of the tube
- $\mathbf{F}_2$: blocking force due to fluid pressure at the end of the tube
- $\mathbf{F}_3$: force of friction slowing down the selected fluid volume along the outer surface
- $\mathbf{F}_4$: force of friction accelerating the selected fluid volume along the inner surface
- $\Delta r$: thickness of the selected fluid volume
- $r$: distance from the axis
- $R$: radius of the tube
- $A_\mathrm{f}$: area of the front surface of the selected volume
- $A(r)$: lateral surface area (area of the side) of the selected volume (different for inner and outer surfaces)

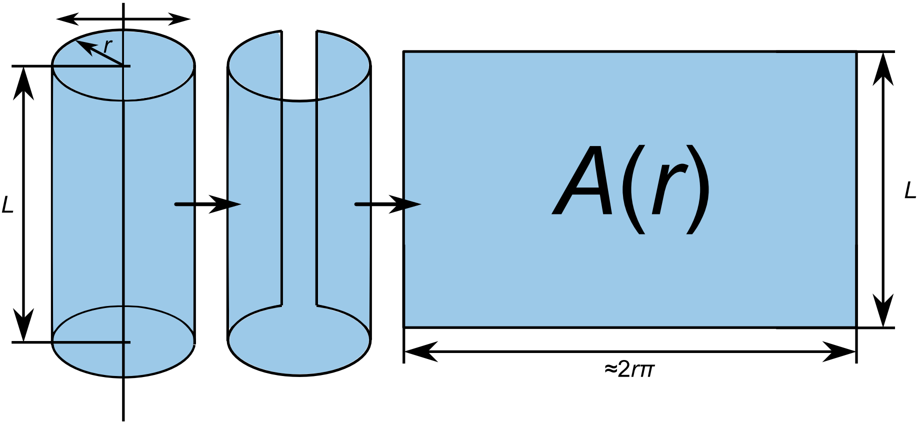
# Properties of the flow

- the tube has a cylindrical symmetry ⇒ fluid layers will have a cylindrical symmetry
- particles at the same distance $r$ from the axis travel with the same velocity
- particles at the wall: do not move due to adhesion to the wall ($v = 0$)
- particles in the axis: maximum speed
- we select a hollow tube of fluid with thickness $\Delta r$ at distance $r$ from the axis
- fluid moving *within* the selected volume: moving faster ⇒ accelerating the selected fluid volume due to friction
- fluid moving *outside* the selected volume: moving slower ⇒ decelerating the selected fluid volume due to friction
- pressure of the rest of the fluid: accelerates the selected volume at one end, decelerates it at the other
- we consider a stationary flow ⇒ acceleration is zero ⇒ the vector sum of all the forces acting on the selected fluid volume is zero

# Front surface area



# Lateral surface area



# Forces acting on the selected fluid volume

## Forces due to pressure

- pressure at the beginning is greater than pressure at the end: $p_1 > p_2 \Rightarrow$ direction of the flow
- $A_\mathrm{f} \approx 2r\pi\Delta r$ (see the figure)
- $F_1 = p_1 A_\mathrm{f} = p_1 2r\pi\Delta r$
- $F_2 = -p_2 A_\mathrm{f} = -p_2 2r\pi\Delta r$

## Forces due to viscosity

- Newton's law of friction: $\tau = \eta\frac{\mathrm{d}v}{\mathrm{d}r} \quad \Rightarrow \quad F = A\eta\frac{\mathrm{d}v}{\mathrm{d}r}$
- $\frac{\mathrm{d}v}{\mathrm{d}r}$ is negative because the velocity decreases with $r$
- lateral surface area at distance $r$ from the axis: $A(r) = L \cdot 2r\pi$ (see the figure)
- force of outer layer (blocking): $F_3 = \eta A(r + \Delta r) \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r+\Delta r}$
- force of inner layer (propelling): $F_4 = -\eta A(r) \left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r}$

# Obtaining the velocity

- we consider a stationary flow ⇒ acceleration is zero ⇒ the vector sum of all the forces acting on the selected fluid volume is zero (Newton's second law)

$$\sum F = F_1 + F_2 + F_3 + F_4 = 0$$

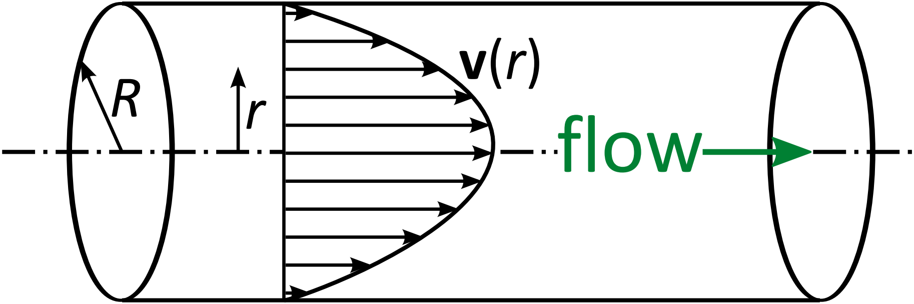
$$p_1 2r\pi\Delta r - p_2 2r\pi\Delta r + \eta A(r+\Delta r)\left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r+\Delta r} - \eta A(r)\left.\frac{\mathrm{d}v}{\mathrm{d}r}\right|_{r} = 0$$

- solving this differential equation yields

$$v(r) = \frac{p_1 - p_2}{4\eta L}\left(R^2 - r^2\right)$$

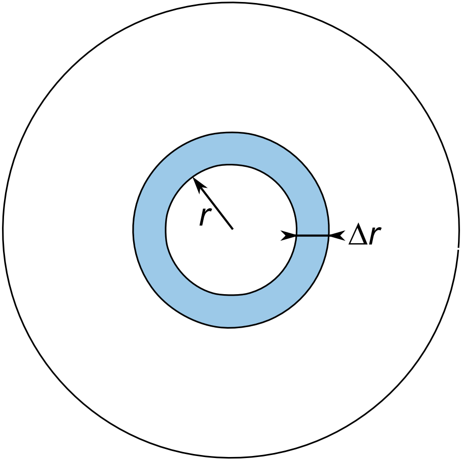
- it is called the **parabolic velocity profile**
- the farther we are from the axis, the less the velocity
- $v(0) = \frac{p_1 - p_2}{4\eta L} R^2$ — maximum value
- $v(R) = 0$

# Parabolic velocity profile

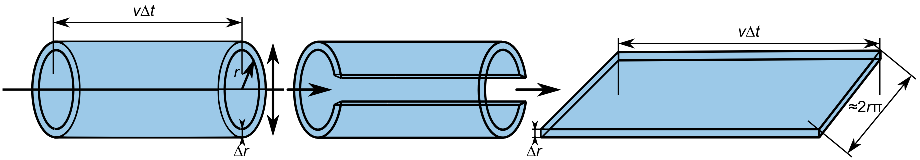


# Obtaining the current

- we now know the velocity, but not the current
- let us consider the *current* flowing through a $\Delta r$ thick ring at distance $r$ from the axis: $I(r)$
- the ring is very narrow $\Rightarrow$ $v(r)$ can be considered as constant within the ring
- here we consider the *volume current:* $I(r) = \frac{\Delta V}{\Delta t}$, where $\Delta V$ is the fluid volume flowing through our selected ring in time $\Delta t$



# Obtaining the volume $\Delta V$

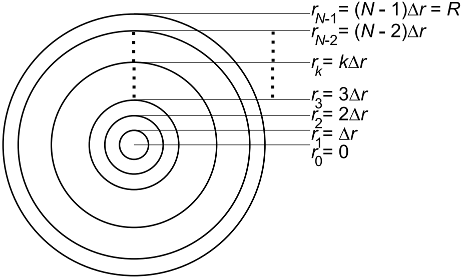



$$\Delta V = 2r\pi \cdot v(r)\Delta t \cdot \Delta r$$

$$I(r) = \frac{\Delta V}{\Delta t} = \frac{2r\pi \cdot v(r)\Delta t \cdot \Delta r}{\Delta t} = 2r\pi \cdot v(r) \cdot \Delta r$$

# Summing the currents

- to obtain the total current $I$, we have to add up the currents $I(r)$ of all the rings constituting the total cross section
- the cross section is broken down into $N$ rings of equal thickness $\Delta r$
- for the $k^{\text{th}}$ ring, the inner radius is $r_k := k\Delta r$
- the current through the $k^{\text{th}}$ ring: $I_k := I(r_k) = 2r_k\pi \cdot v(r_k) \cdot \Delta r$
- the total current: $I \approx \sum_{k=0}^{N-1} I_k = \sum_{k=0}^{N-1} 2r_k\pi \cdot v(r_k) \cdot \Delta r$



# Hagen–Poiseuille equation

- exact solution obtained through integration

$$I = \lim_{\Delta r \to 0} \sum_{k=0}^{N-1} 2 r_k \pi \cdot v(r_k) \cdot \Delta r = \int_0^R 2 r \pi \cdot v(r) \, \mathrm{d} r$$

$$I = \int_0^R 2 r \pi \cdot v(r) \, \mathrm{d} r = \int_0^R 2 r \pi \cdot \frac{\Delta p}{4 \eta L} \left( R^2 - r^2 \right) \, \mathrm{d} r = \frac{\pi}{2 \eta L} \cdot \Delta p \cdot \int_0^R \left( r R^2 - r^3 \right) \, \mathrm{d} r$$

$$I = \frac{\pi}{2 \eta L} \cdot \Delta p \cdot \left( R^2 \int_0^R r \, \mathrm{d} r - \int_0^R r^3 \, \mathrm{d} r \right) = \frac{\pi}{2 \eta L} \cdot \Delta p \cdot \left( R^2 \left[ \frac{r^2}{2} \right]_0^R - \left[ \frac{r^4}{4} \right]_0^R \right)$$

$$I = \frac{\pi}{2 \eta L} \cdot \Delta p \cdot \left\{ R^2 \left( \frac{R^2}{2} - 0 \right) - \left( \frac{R^4}{4} - 0 \right) \right\} = \frac{\pi}{2 \eta L} \cdot \Delta p \frac{R^4}{4}$$

$$I = \frac{\pi}{8 \eta L} \cdot \Delta p \cdot R^4$$

- this is the **Hagen–Poiseuille equation**

# Medical examples

## Muscles in use (eg, sports)

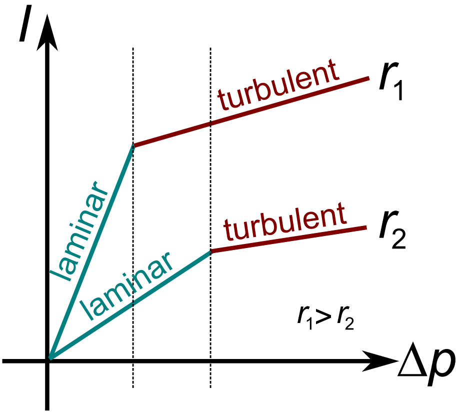
- oxygen demand increases
- more blood (higher current) needed to supply this oxygen
- blood vessels dilate (become wider)
- $I \propto R^4$
- the current through the dilated blood vessels increases drastically

## *Arteriosclerosis*

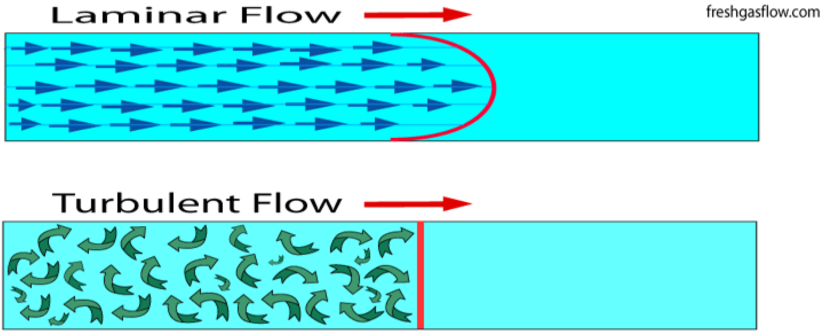
- blood vessels become rigid and *narrow*
- narrower tubes $\Rightarrow$ $R$ drops
- $I \propto R^4 \Rightarrow I$ drops drastically
- much higher pressure needed to maintain the same current
- the heart may not sustain the increased load

# Laminar and turbulent flow

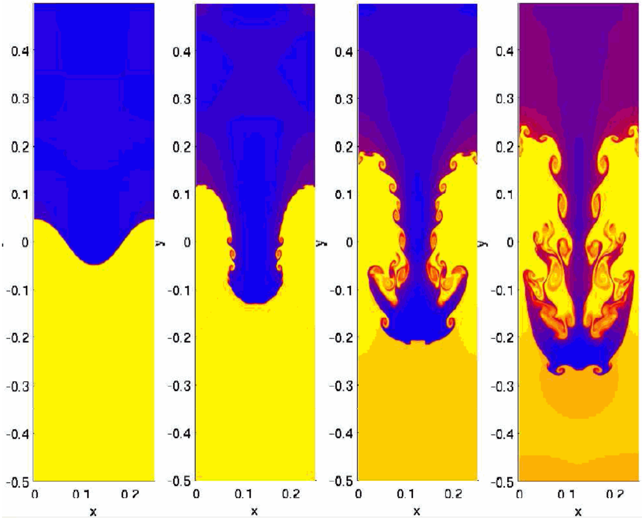
- **laminar flow:** a fluid flows in parallel layers, with no disruption between the layers
- **turbulent flow:** eddies form, flow properties become chaotic
- turbulent flow carries less volume for the same pressure difference: eddies reverse the direction of the flow and energy is dissipated



# Laminar and turbulent flow



# Laminar and turbulent flow



# Laminar and turbulent flow

# Laminar and turbulent flow

# Reynolds number

- Is the flow laminar or turbulent? $\Leftarrow$ *Reynolds number*

$$\mathrm{Re} = \frac{\rho v L}{\eta}$$

- $\rho$: density of the fluid
- $v$: velocity of the flow
- $L$: characteristic length (eg, radius of the tube)
- $\eta$: viscosity of the fluid

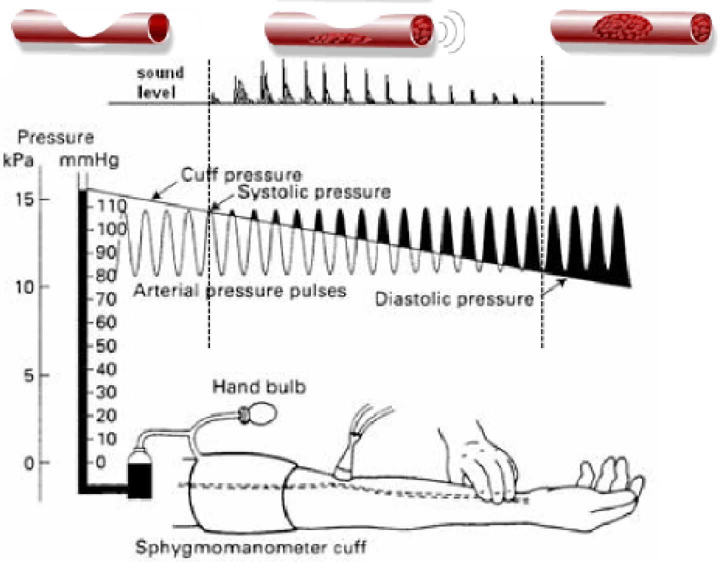
- Re: dimensionless empirical number
- How is it used to predict turbulence? $\Rightarrow$ *transition Reynolds numbers*
  - $\mathrm{Re}_{\mathrm{flow}} < \mathrm{Re}_1$: laminar flow
  - $\mathrm{Re}_1 \leq \mathrm{Re}_{\mathrm{flow}} < \mathrm{Re}_2$: transition region — both laminar and turbulent flow are possible
  - $\mathrm{Re}_2 \leq \mathrm{Re}_{\mathrm{flow}}$: turbulent flow
- transition Reynolds numbers are determined experimentally (eg, wind tunnels)
- transition Reynolds number values are similar for different fluids, they mostly depend on flow geometry

# Example: blood pressure measurement

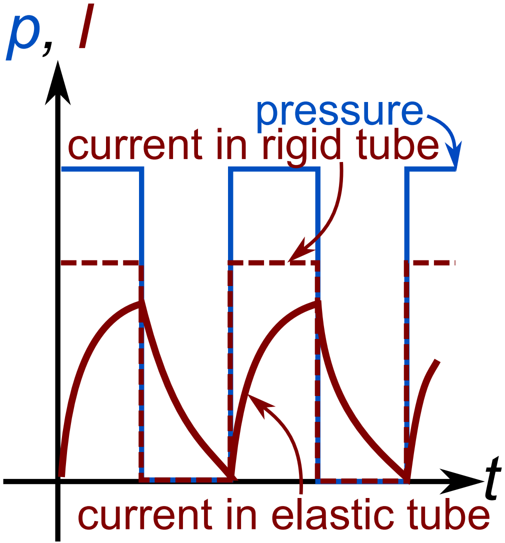
# Example: blood pressure measurement

- the cuff (called the *sphygmomanometer cuff*) is inflated
- the cuff squeezes the artery ⇒ radius of the artery is reduced
- equation of continuity: flow velocity will increase through a narrower cross section
- increased velocity ⇒ Reynolds number increases ⇒ turbulent flow
- turbulent flow is noisy *(Korotkoff sounds)*
- these sounds are audible through the stethoscope

# Example: blood pressure measurement



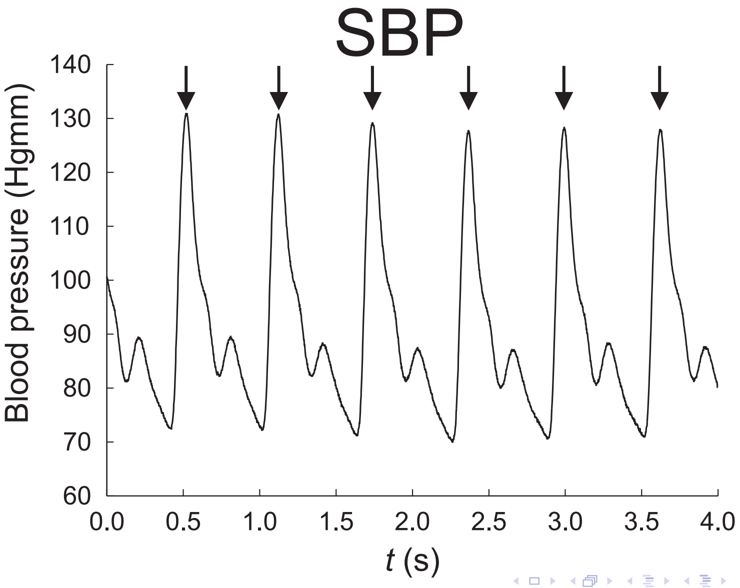
# Pulsatile flow



# Pulsatile flow

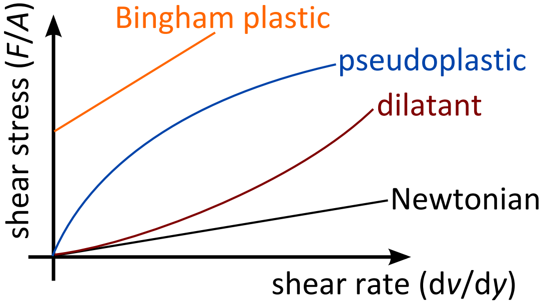
- eg, heart: operates in cycles of contraction and relaxation ⇒ *pulsatile flow*
- simple model: a tap opened and closed periodically
- elastic tube
  - high pressure: the walls of the tube expand and store elastic energy; the kinetic energy of the flow is converted to elastic energy
  - low pressure: the walls of the tube relax from their expanded state; their elastic energy is converted back to kinetic energy
  - this way, flow is maintained even when the pressure difference is low
  - physiological significance: the elasticity of veins makes blood pressure more stable
- rigid tube
  - no expansion or relaxation possible
  - lost mechanical energy is converted to heat

# Blood pressure



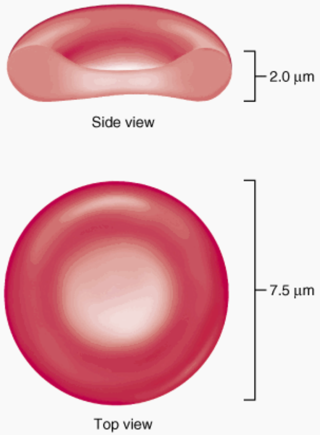
# Types of non-Newtonian fluids

- Newtonian fluids
  - obey Newton's law of friction
  - their viscosity does not depend on shear rate
- non-Newtonian fluids
  - do *not* obey Newton's law of friction
  - their viscosity depends on shear rate, and in many cases, on the history of the fluid



# Flow properties of blood

- volume of blood: 5–6 litres
- cellular components of blood:
  - erythrocytes (red blood cells)
  - leukocytes (white blood cells)
  - platelets
- the diameter of erythrocytes is greater than that of capillaries ⇒ erythrocytes can only squeeze through capillaries due to their elasticity
- **bolus flow**: erythrocytes block normal flow, blood plasma between them is forced to rotate ⇒ better circulation
- anomalies: sickle anaemia — reduced membrane elasticity of erythrocytes ⇒ being trapped (ageing of erythrocytes also reduces elasticity)



# Bolus flow